

6.

CTNT 2024

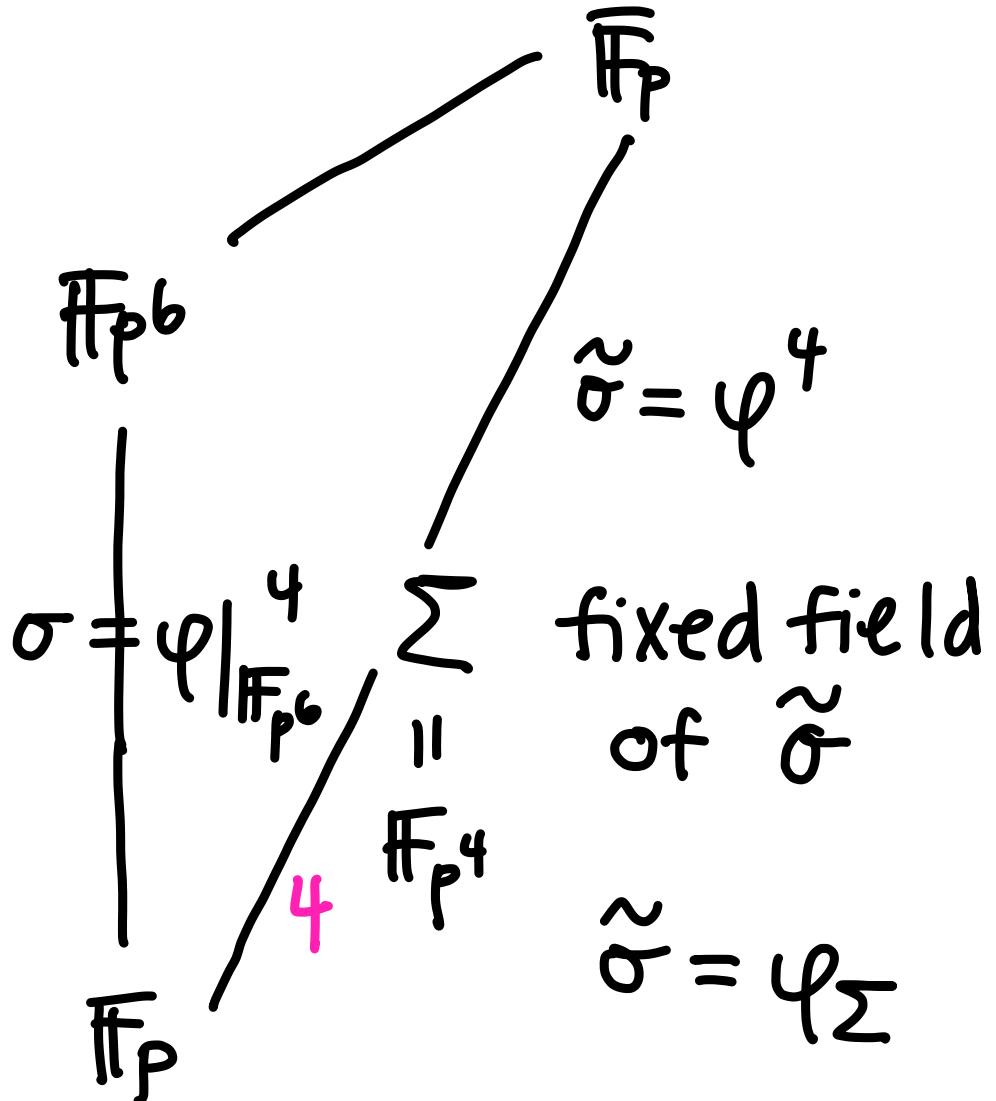
Connecticut Summer School in Number Theory

Class Field Theory

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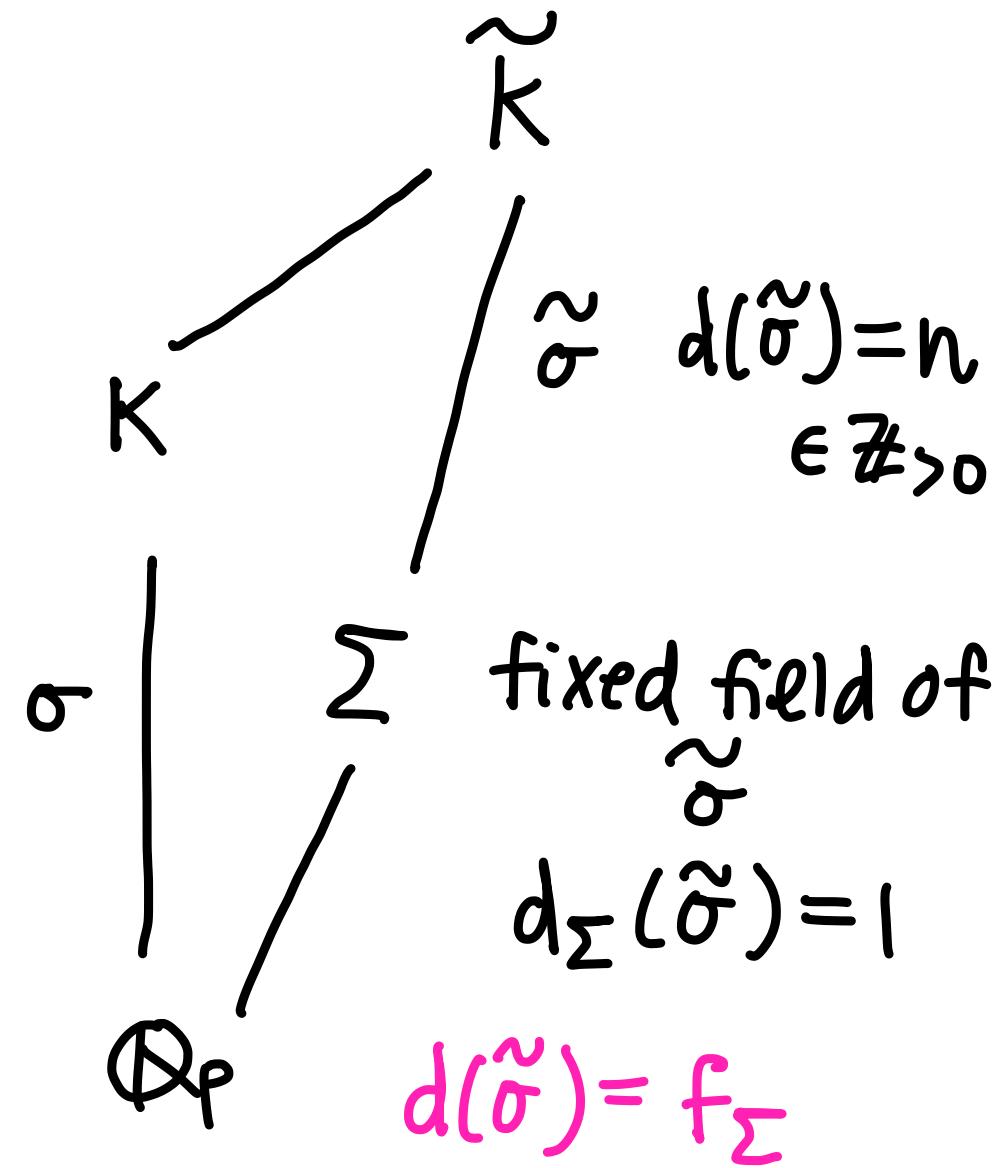
finite fields

$$d: \text{Gal}(\bar{\mathbb{F}_p}/\mathbb{F}_p) \xrightarrow{\sim} \hat{\mathbb{Z}}_4$$



over \mathbb{Q}_p

$$d: \text{Gal}(\bar{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \hat{\mathbb{Z}}$$



$$I = \ker d = \text{Gal}(\bar{\mathbb{Q}_p}/\tilde{\mathbb{Q}_p})$$

finite Galois extension K/\mathbb{Q}_p

\mathbb{Z}_p unique prime ideal (p)

\mathcal{O}_K also

$$p\mathcal{O}_K = \mathfrak{p}^e$$

$p = (\pi_K)$ π_K is a prime in K

$$\sigma \in \text{Gal}(K/\mathbb{Q}_p) \quad \sigma(p) = \mathfrak{p} \quad \sigma(\pi_K) = u_\sigma \pi_K$$

$$\sigma \in \text{Gal}(K/\mathbb{Q}_p) \quad \sigma(\beta) = \beta \quad \sigma(\pi_K) = u_0 \pi_K$$

$x \in K$ is of the form $x = v \pi_K^m$ $v \in O_K^\times$

$$\left[x \in \mathbb{Q}_p \quad x = \sum_{i=n}^{\infty} a_i p^i = p^n (a_n + a_{n+1} p + \dots) \right]$$

$a_n \neq 0$

\mathbb{Z}_p^\times

$$v_p(x) = m$$

FOR $x \in K$, define

$$N_{K/\mathbb{Q}_p}(x) = \prod_{\sigma \in \text{Gal}(K/\mathbb{Q}_p)} \sigma(x)$$

if $x = v \pi_K^m$

$$\sigma(\pi_K) = u_\sigma \pi_K$$

$$= N_{K/\mathbb{Q}_p}(v) N_{K/\mathbb{Q}_p}(\pi_K)^m$$

$$[K:\mathbb{Q}_p] = f_K e_K$$

$$= \boxed{N_{K/\mathbb{Q}_p}(v) \prod_{\sigma} u_\sigma^m}$$

unit

$$P = \pi_K^{e_K}$$

$$\pi_K^{mf_K e_K}$$

$$v_p(N_{K/\mathbb{Q}_p}(x)) = v_p(\underline{\pi_K}^{mef_K})$$

$$= v_p(p^{mf_K})$$

$$= mf_K$$

$$v_p(N_{K/\mathbb{Q}_p}(x)) = f_K v_p(x)$$

$$v_p(N_{K/\mathbb{Q}_p}(\pi_K)) = f_K v_p(\pi_K) = f_K$$

Reciprocity map

(Hypotheses) There is a group hom

$$r_{K/\mathbb{Q}_p} : \text{Gal}(K/\mathbb{Q}_p) \rightarrow \mathbb{Q}_p^\times / N_{K/\mathbb{Q}_p}(K^\times)$$

$$\sigma \mapsto N_{\Sigma/\mathbb{Q}_p}(\pi_\Sigma) \bmod N_{K/\mathbb{Q}_p}(K^\times)$$



$$d(\tilde{\sigma}) = f_\Sigma = v_p(N_{\Sigma/\mathbb{Q}_p}(\pi_\Sigma))$$

What are the hypotheses?

For every finite unramified extension
 L/K (extensions of \mathbb{Q}_p)

- every unit in \mathcal{O}_K is the norm of a unit in \mathcal{O}_L
- if a unit u in \mathcal{O}_L has norm 1, then there is another unit v in \mathcal{O}_L such that
$$u = \frac{\varphi_{L/K}(v)}{v}$$

Reciprocity Law

Under more hypotheses

for all K/\mathbb{Q}_p
finite

$$r_{K/\mathbb{Q}_p} : \text{Gal}(K/\mathbb{Q}_p)^{\text{ab}} \xrightarrow{\sim} \frac{\mathbb{Q}_p^\times}{N_{K/\mathbb{Q}_p}(K^\times)}$$

Classify abelian extensions of \mathbb{Q}_p

There is a 1-1 correspondence btw

finite abelian
extensions of \mathbb{Q}_p

open subgps of
finite index in \mathbb{Q}_p^\times

$$K \mapsto N_{K/\mathbb{Q}_p}(K^\times)$$

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Fact every open subgp of \mathbb{Q}_p^\times of finite index contains

$$(p^f)^\times \cup_{\mathbb{Q}_p}^{(n)}$$

$b \in \mathbb{Z}_p$

elements of the form $1 + p^n \cdot b$

Fact every open subgp of \mathbb{Q}_p^\times of finite index contains

$$(p^f) \times U_{\mathbb{Q}_p}^{(n)}$$

be \mathbb{Z}_p^\times

elements of the form $1 + p^n \cdot b$

$$N_{\mathbb{Q}_p(\mathcal{I}_{p^n})/\mathbb{Q}_p}(\mathbb{Q}_p(\mathcal{I}_{p^n})^\times) = (p) \times U_{\mathbb{Q}_p}^{(n)}$$

$$N_{\mathbb{Q}_p(\mathcal{I}_r)/\mathbb{Q}_p}(\mathbb{Q}_p(\mathcal{I}_r)^\times) = (p^f) \times U_{\mathbb{Q}_p}^{(1)}$$

$\gcd(r, p) = 1$ \nexists smallest pos. Int. s.t.
 $p^f \equiv 1 \pmod r$

\cup open finite index $\supseteq N_{\mathbb{Q}_p(S)/\mathbb{Q}_p}(\mathbb{Q}_p(S)^\times)$

$N_{K/\mathbb{Q}_p}(K^\times)$

$K \subseteq \mathbb{Q}_p(S)$

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Nun einem bei
G der Or
gehörige

tzung) zu
ment aus
der zu g^σ

Ist $\psi_i(\tau)$ ein Charakter (also gewöhnlicher Abel'scher Gruppencharakter) von $g^\sigma(i = 1, 2 \dots m(\sigma))$, wo $i = 1$ der Hauptcharakter sei), so bilden wir in Ω die L -Reihe in bezug auf K :

$$L(s, \psi_i^{(\sigma)}):$$

Wegen (15) und (7) gilt, wenn wir noch den Index σ zur Unterscheidung anbringen:

...That's all for now ...

$$(23) \quad L(s, \psi_i^{(\sigma)}) = \prod_{\nu=1}^r (L(s, \chi_\nu))^{r_{i\nu}^{(\sigma)}} \quad (i = 1, 2, \dots m(\sigma)).$$

Nach der gemachten Voraussetzung steht links eine gewöhnliche Abel'sche L -Reihe. Deshalb gestattet (23) die Fortsetzbarkeit unserer Funktionen zu beweisen. Zunächst ist $L_i(s, \chi^1) = f_i(s)$. Für sie ist also die Fortsetzbarkeit bewiesen.