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Connecticut Summer School in Number Theory

Class Field Theory

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finite fields

$d: \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \xrightarrow{\sim} \hat{\mathbb{Z}}$

$\tilde{\sigma} \longmapsto 4$

$$\begin{tikzcd}
 & & \overline{\mathbb{F}}_p \arrow[dll, dash] \arrow[dl, dash, "\tilde{\sigma} = \varphi^4"] \\
\mathbb{F}_{p^6} \arrow[d, dash, "\sigma = \varphi^4|_{\mathbb{F}_{p^6}}"'] & \Sigma = \mathbb{F}_{p^4} & \\
\mathbb{F}_p \arrow[ur, dash, "4"] & &
\end{tikzcd}$$

Σ fixed field of $\tilde{\sigma}$

$\tilde{\sigma} = \varphi_\Sigma$

over $\mathbb{Q}_p$

$d: \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \hat{\mathbb{Z}}$

$$\begin{tikzcd}
 & & \tilde{K} \arrow[dll, dash] \arrow[d, dash, "\tilde{\sigma}"] \\
K \arrow[d, dash, "\sigma"'] & & \Sigma \\
\mathbb{Q}_p \arrow[urr, dash] & &
\end{tikzcd}$$

$d(\tilde{\sigma}) = n \in \mathbb{Z}_{>0}$

Σ fixed field of $\tilde{\sigma}$

$d_\Sigma(\tilde{\sigma}) = 1$

$d(\tilde{\sigma}) = f_\Sigma$

$I = \ker d = \mathrm{Gal}(\bar{\mathbb{Q}}_p / \tilde{\mathbb{Q}}_p)$

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finite Galois extension $K/\mathbb{Q}_p$

$\mathbb{Z}_p$ unique prime ideal (p)

$\mathcal{O}_K$ also $\mathfrak{p}$ $\quad p\mathcal{O}_K = \mathfrak{p}^e$

$\mathfrak{p} = (\pi_K)$ $\quad \pi_K$ is a prime in K

$\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$ $\quad \sigma(\mathfrak{p}) = \mathfrak{p}$ $\quad \sigma(\pi_K) = u_\sigma \pi_K$

$\sigma \in Gal(K/\mathbb{Q}_p) \quad \sigma(p) = p \quad \sigma(\pi_K) = u_\sigma \pi_K$

$x \in K$ is of the form $\quad x = v\pi_K^m \quad v \in \mathcal{O}_K^\times$

$$\left[x \in \mathbb{Q}_p \quad x = \sum_{i=n}^{\infty} a_i p^i = p^n \underbrace{(a_n + a_{n+1}p + \dots)}_{\mathbb{Z}_p^\times} \right]$$

$a_n \neq 0$

$v_p(x) = m$

For $x \in K$, define

$$N_{K/\mathbb{Q}_p}(x) = \prod_{\sigma \in Gal(K/\mathbb{Q}_p)} \sigma(x)$$

$\sigma(\pi_K) = u_\sigma \pi_K$

if $x = v\pi_K^m$

$$= N_{K/\mathbb{Q}_p}(v)\, N_{K/\mathbb{Q}_p}(\pi_K)^m$$

$[K:\mathbb{Q}_p] = f_K e_K$

$$= \underbrace{\boxed{N_{K/\mathbb{Q}_p}(v) \prod_\sigma u_\sigma^m}}_{\text{unit}} \boxed{\pi_K^{m[K:\mathbb{Q}_p]} \quad \pi_K^{mf_Ke_K}}$$

$p = \pi_K^{e_K}$

$$v_p\left(N_{K/\mathbb{Q}_p}(x)\right) = v_p\left(\underline{\pi_K}^{m\underline{e_K}f_K}\right)$$

$$= v_p\left(p^{mf_K}\right)$$

$$= mf_K$$

$$v_p\left(N_{K/\mathbb{Q}_p}(x)\right) = f_K\, v_p(x)$$

$$v_p\left(N_{K/\mathbb{Q}_p}(\pi_K)\right) = f_K\, v_p(\pi_K) = f_K$$

Reciprocity map

(Hypotheses) There is a group hom

$$r_{K/\mathbb{Q}_p} : \mathrm{Gal}(K/\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times / N_{K/\mathbb{Q}_p}(K^\times)$$

$$\sigma \longmapsto N_{\Sigma/\mathbb{Q}_p}(\pi_\Sigma) \bmod N_{K/\mathbb{Q}_p}(K^\times)$$

$$d(\tilde{\sigma}) = f_\Sigma = v_p\left(N_{\Sigma/\mathbb{Q}_p}(\pi_\Sigma)\right)$$

What are the hypotheses?

For every finite unramified extension L/K (extensions of $\mathbb{Q}_p$)

- every unit in $\mathcal{O}_K$ is the norm of a unit in $\mathcal{O}_L$

- if a unit u in $\mathcal{O}_L$ has norm 1, then there is another unit, v in $\mathcal{O}_L$ such that

$$u = \frac{\varphi_{L/K}(v)}{v}$$

Reciprocity Law

Under more hypotheses

for all $K/\mathbb{Q}_p$ finite

$$r_{K/\mathbb{Q}_p} : \mathrm{Gal}(K/\mathbb{Q}_p)^{ab} \xrightarrow{\sim} \mathbb{Q}_p^\times / N_{K/\mathbb{Q}_p}(K^\times)$$

Classify abelian extensions of $\mathbb{Q}_p$

There is a 1-1 correspondence btw

finite abelian extensions of $\mathbb{Q}_p$	open subgps of finite index in $\mathbb{Q}_p^\times$
K $\longmapsto$	$N_{K/\mathbb{Q}_p}(K^\times)$

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Fact every open subgp of $\mathbb{Q}_p^\times$ of finite index contains

$$(p^f) \times \underbrace{U_{\mathbb{Q}_p}^{(n)}}_{\text{elements of the form } 1+p^n \cdot b}$$

$b \in \mathbb{Z}_p$

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$$N_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}\left(\mathbb{Q}_p(\zeta_{p^n})^\times\right) = (p) \times U_{\mathbb{Q}_p}^{(n)}$$

$$N_{\mathbb{Q}_p(\zeta_r)/\mathbb{Q}_p}\left(\mathbb{Q}_p(\zeta_r)^\times\right) = \left(p^f\right) \times U_{\mathbb{Q}_p}^{(1)}$$

$\gcd(r,p)=1$

f smallest pos. int. s.t. $p^f \equiv 1 \bmod r$

$\mathcal{U}$ open finite index $\supseteq N_{\mathbb{Q}_p(\zeta)/\mathbb{Q}_p}(\mathbb{Q}_p(\zeta)^\times)$

$\|$

$N_{K/\mathbb{Q}_p}(K^\times)$

$$K \subseteq \mathbb{Q}_p(\zeta)$$

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... That's all for now ...