G-Valued Crystalline Deformation Rings in the Fontaine-Laffaille Range

joint work with Brandon Levin

Ex: $f(q) = q\prod(1-q^n)^2(1-q^{11n})^2 = \sum a_n(f)q^n = q - 2q^2 - q^3 + \dots$

$E: y^2 + y = x^3 - x^2$ $\quad a_p(E) = p + 1 - \#E(\mathbb{F}_p)$

p	2	3	5	7	11	13	17	19
$a_p(f)$	-2	-1	1	-2	1	4	-2	0
$a_p(E)$	-2	-1	1	-2	1	4	-2	0

E is elliptic curve conductor 11

f is q-expansion of modular form wt 2, level 11

E is modular: $a_p(E) = a_p(f)$

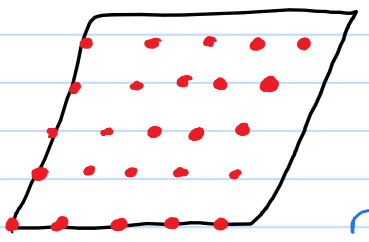
Further Observation: $a_p(f) \equiv p+1 \mod 5$ $\quad p \neq 11$

Explanations: congruence of modular forms... or

E has a rational 5-torsion point $\Rightarrow 5 \mid \#E(\mathbb{F}_p)$

$a_p(E) = p+1-\#E(\mathbb{F}_p)$
$\equiv p+1 \mod 5$

Another Perspective:
Galois Representations



5-torsion of E

$E[5] \simeq (\mathbb{Z}/5\mathbb{Z})^2$

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

rational — action fixes them

ℓ prime. Similarly, $E[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^2 \circlearrowleft \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$T_\ell E \simeq \mathbb{Z}_\ell^2$

$\rho_{E,\ell}: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_\ell)$

Key: char. poly of $\rho_{E,\ell}(\mathrm{Frob}_p)$ is $x^2 - a_p(E)x + p$ (almost all p)

Our Example: $\bar{\rho}_{E,5}(\mathrm{Frob}_p) = \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix} \mod 5$

traces: $a_p(E) \equiv 1 + p \mod 5$

Likewise, the cusp form f has Galois representation:

$\rho_{f,\ell}: \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_\ell)$

Frob_p has char. poly $x^2 - a_p(f)x + p$ (almost all p)

E modular means $\rho_{E,\ell} \simeq \rho_{f,\ell}$

Example $E': y^2+xy+y = x^3+x+1$ elliptic curve/$\mathbb{Q}$ Conductor 38

p	2	3	5	7	11	13	17	19
$a_p(E)$	-2	-1	1	-2	1	4	-2	0
$a_p(E')$	1	-1	-4	3	2	-1	3	-1

Observations: $a_p(E') \equiv p+1 \equiv a_p(E) \bmod 5$ $\quad p \neq 2, 11, 19$

Explanations: E' has rational 5-torsion point

Better Explanation: $\rho_{E,5} \equiv \rho_{E',5} \bmod 5$

Question: Is E' modular?

1) Yes, look for cusp form (level 38)
2) Yes, Modularity lifting theorems

Key Tool in Modern Number Theory:
understand congruences between Galois representations

Given $\bar{\rho}$, $\quad$ $\rho \nearrow GL_2(A) \searrow$ $\quad$ A like $\mathbb{Z}/p^n\mathbb{Z}$, $\mathbb{Z}_p$

$$\mathrm{Gal}(\bar{K}/K) \xrightarrow{\bar{\rho}} GL_2(\mathbb{Z}/p\mathbb{Z})$$

field, p-adic

Study moduli space of lifts $R_{\bar{\rho}}^{\square}$ $\leftarrow$ adding extra cond.

Also want to replace GL_2 with reductive group: GL_n, SO_n, Sp_{2n}, E_8

How to study:

K number field	work locally; restrict to decomp. gps
K ℓ-adic field $\ell \neq p$	hope ℓ-adic vs. p-adic top restricts options
K p-adic field	Way too many $\mathrm{Gal}(\bar{K}/K) \to G(\mathbb{Z}_p)$

p-adic Hodge theory: identify which "are geometric"

$$\{\text{nice Galois reps}\} \longleftrightarrow \{\text{Semilinear algebra objects}\}$$

dependent on coefficients

Today: crystalline representations

Example: Cyclotomic Character $\chi: \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$
σ $\qquad \sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi(\sigma)}$

Example:
$$\begin{bmatrix} \chi^{r_1} & * & * \\ 0 & \chi^{r_2} & * \\ 0 & 0 & \chi^{r_3} \ddots \end{bmatrix}$$
$r_1 - r_2 > 1$
$r_2 - r_3 > 1$ automatically crystalline
etc

Example: A Abelian Variety / $\mathbb{Q}_p$. good reduction

$\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \circlearrowright T_p A$, crystalline + symplectic

Invariant of Representation: Hodge-Tate weights $\{r_i\}$

p-adic Hodge type: $\mu: \mathbb{G}_m \to G$

Crystalline $\rho: \mathrm{Gal}(\bar{K}/K) \to GL_n(\mathbb{Q}_p)$ $\quad$ OK

"Crystalline" $\rho: \mathrm{Gal}(\bar{K}/K) \to GL_n(\mathbb{Z}/p^n\mathbb{Z})$???

OK if $r_1 - r_n < p-1$ via Fontaine-Laffaille theory

Definition: $\mu: \mathbb{G}_m \to G$ 1) Fontaine-Laffaille if $\max_{\alpha \in \Phi_G} \langle \alpha, \mu \rangle < p-1$

2) Strongly FL $\qquad$ Same $< \frac{p-1}{2}$

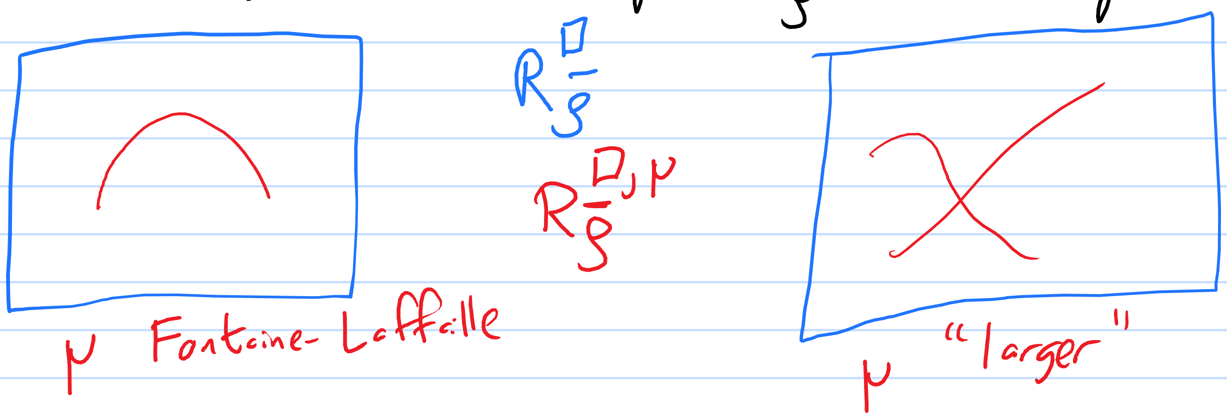
Note: $\underbrace{e_i - e_j}_{\text{root for } GL_n}$ pairs with $\begin{pmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{r_2} & 0 \\ 0 & 0 & t^{r_3} \ddots \end{pmatrix}$ as $r_i - r_j$

Theorem: $\mathbb{F}$ finite field of char p
G split connected reductive group $/\mathbb{F}$ $\quad p \nmid \#\pi_1(G)$
Fix continuous $\bar{\rho}: Gal(\bar{K}/K) \to G(\mathbb{F})$ w/ type μ

If K is unramified over $\mathbb{Q}_p$ and either
a) μ is strongly Fontaine-Laffaille or
b) μ is Fontaine-Laffaille and G simply connected

then the universal crystalline def ring $R_{\bar{\rho}}^{\square,\mu}$ is formally smooth over $W(\mathbb{F})$
(if non-empty)

Remark: This is when we expect $R_{\bar{\rho}}^{\square,\mu}$ to be "simple"



Remark: Some cases known: Ramakrisha: flat, GL_2
Clozel-Harris-Taylor GL_n using Fontaine-Laffaille theory
Patrikis, B extend to SO_n, Sp_{2n}
Our proof independent of FL theory

Remark: We expect $R_{\bar{\rho}}^{\square,\mu}$ to be non-empty for correct μ's
crystalline lifts

Remark: Hypothesis that G is simply connected or μ strongly FL $\Big\}$ necessary

PGL_2 example:

$$\begin{array}{ccc} Gal(\bar{L}/L) & \xrightarrow{\bar{\rho}} & GL_2(k) \\ & \searrow^{\bar{\rho}'} & \downarrow \\ & & PGL_2(k) \end{array}$$

$\bar{\rho} \simeq 1 \oplus \bar{\chi}^{p-1/2}$
$\simeq \bar{\chi}^{p-1} \oplus \bar{\chi}^{p-1/2}$

two plausible μ:

$t \mapsto \begin{pmatrix} t^0 & 0 \\ 0 & t^{p-1/2} \end{pmatrix}$ $\qquad$ $t \mapsto \begin{pmatrix} t^{p-1} & 0 \\ 0 & t^{p-1/2} \end{pmatrix}$

For GL_2, the $R_{\bar{\rho}}^{\square,\mu}$ both smooth
Can't distinguish in PGL_2, so expect two intersecting components

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Proof Strategy: Can't use Fontaine-Laffaille theory

limited compatibility w/ $\otimes$

Outline:

all use FL hyp.

- relate $R_{\bar{\rho}}^{\square,\mu}$ to deformations of Kisin modules
- understand when a Kisin module "looks crystalline"
- compare with a smooth locus in affine Grassmanian.

Inspired by work of Le, Le Hung, Levin, and Morra

$\square$