Semistable models of hyperelliptic curves over residue characteristic 2

Jeff Yelton (Emory University)
joint work with
Leonardo Fiore

University of Connecticut (virtual)
June 13th, 2020

Hyperelliptic curves over local fields

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where $f(x) \in K[x]$ is a polynomial of degree $d \geq 3$ and roots $a_1, ..., a_d \in \bar{K}$. (If $d = 3$ then it is also an **elliptic curve**.)

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Since the characteristic $\neq 2$ the singular points on C are of the form $(a, 0)$ where $f(a) = f'(a) = 0$. Therefore, the *smoothness* property means that the roots a_i of f are all distinct.

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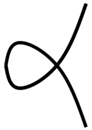

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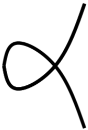

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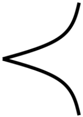

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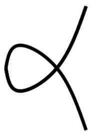

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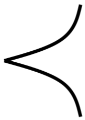

On the other hand, if $p = 2$, the reduced curve $\bar{C} : y^2 = \prod_{i=1}^{d}(x - \bar{a}_i)$ *always* has a singular point and every singular point is a cusp!

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Theorem (Dokchitser-Dokchitser-Maistret-Morgan)

When $p \neq 2$, the reduction type of C^{ss} is determined entirely by how the roots "cluster".

Reductions of genus-2 curves when $p \neq 2$

For genus 2, $d = 5$, and $p \neq 2$, here are the main possibilities:

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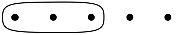
three of a kind 		

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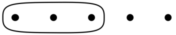
three of a kind 		
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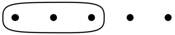
three of a kind 		
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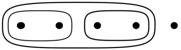
two pairs & four of a kind 		

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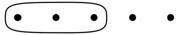
three of a kind 		
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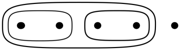
two pairs & four of a kind 		

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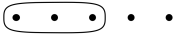
three of a kind 		
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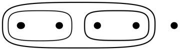
two pairs & four of a kind 		

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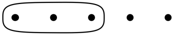
three of a kind 		
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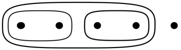
two pairs & four of a kind 		

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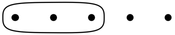
three of a kind 		
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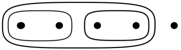
two pairs & four of a kind 		

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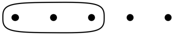
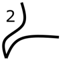
three of a kind 		
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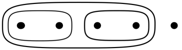
two pairs & four of a kind 		

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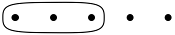
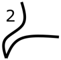
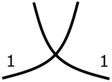
three of a kind 		
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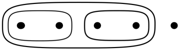
two pairs & four of a kind 		

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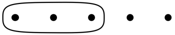
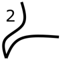
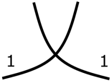
three of a kind 		
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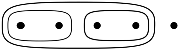
two pairs & four of a kind 		

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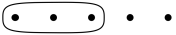
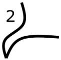
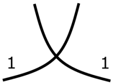
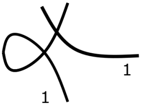
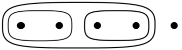
three of a kind 		
full house 		
two pairs & four of a kind 		

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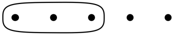
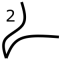
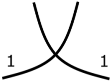
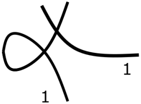
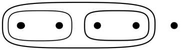
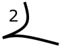
three of a kind 		
full house 		
two pairs & four of a kind 		

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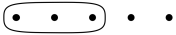
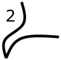
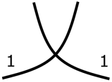
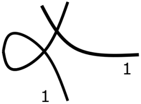
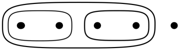
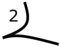
three of a kind 		
full house 		
two pairs & four of a kind 		

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With clever enough choices of α, β for each such substitution, the above equation(s) together reduce to a semistable curve.

Semistable reduction types for elliptic curves ($p = 2$)

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As seen above, we can convert any elliptic curve to one with (at most) one cluster $\{a_1, a_2\}$. Let $v : K^\times \to \mathbb{Q}$ be the valuation on K normalized so that $v(2) = 1$ and let $m = v(a_2 - a_1) \geq 0$. There are two cases:

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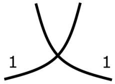
Corollary

In this setting, an elliptic curve has potentially good reduction iff $0 \leq m \leq 4$.

Semistable reduction types for genus 2 ($p = 2$)

cluster picture	reduction type of C^{ss}
no cluster 	
pair 	
two pairs 	

Semistable reduction types for genus 2 ($p = 2$)

cluster picture	reduction type of C^{ss}
no cluster 	(two cases)  or 
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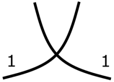
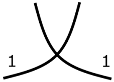
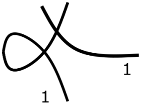
cluster picture	reduction type of C^{ss}
no cluster • • • • •	(two cases)  or 
pair (• •) • • • ($m :=$ valuation of difference between roots in pair)	$0 \le m \le \frac{8}{3}$  or  $\frac{8}{3} < m \le 4$ $m > 4$  or 
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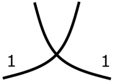
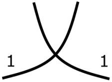
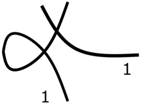
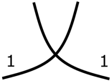
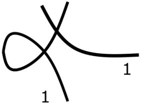
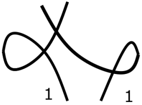
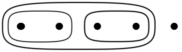
cluster picture	reduction type of C^{ss}
no cluster • • • • •	(two cases)  or 
pair (• •) • • • ($m :=$ valuation of difference between roots in pair)	$0 \leq m \leq \frac{8}{3}$  or  $\frac{8}{3} < m \leq 4$ $m > 4$  or 
two pairs (• •) (• •) • ($m_1, m_2 :=$ valuations of differences between roots in each pair)	two curves C_1, C_2, each with a node iff $m_i > 4$  or  or 

Semistable reduction types of hyperelliptic curves of genus 2 ($p = 2$)

cluster picture	reduction type of C^{ss}
three of a kind 	
full house 	
two pairs and four of a kind 	

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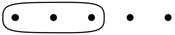
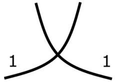
three of a kind 	
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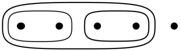
two pairs and four of a kind 	

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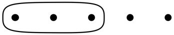
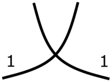
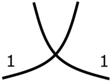
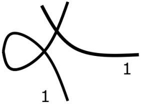
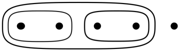
three of a kind 	
full house  ($m :=$ valuation of difference between roots in pair)	$0 \leq m \leq 4$  $m > 4$ 
two pairs and four of a kind 	

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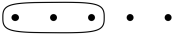
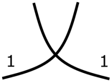
<table>
<tr><th>cluster picture</th><th>reduction type of C^{ss}</th></tr>
<tr><td>three of a kind
</td><td>
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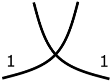
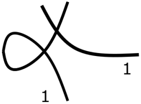
($m :=$ valuation of difference between roots in pair)</td><td>$0 \leq m \leq 4$ 

$m > 4$ 
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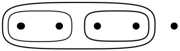</td><td>if evaluations of differences are $\gg 0$

</td></tr>
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