The Ceresa class and hyperelliptic curves

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The Ceresa cycle – definition

Given a smooth projective curve C over a characteristic 0 field K with a rational point $P \in C(K)$, there are two embeddings $C \hookrightarrow \text{Jac}(C)$:

$$\theta_P : Q \mapsto [Q - P], \text{ and } \overline{\theta}_P = [-1] \circ \theta_P : Q \mapsto [P - Q].$$

The **Ceresa cycle** associated to the pair (C, P) is the algebraic cycle

$$\theta_P(C) - \overline{\theta}_P(C) \in CH_1(\text{Jac}(C)).$$

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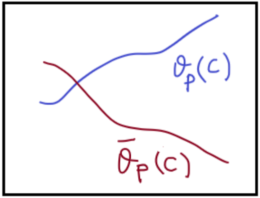

The Ceresa cycle – homologically trivial

Take the case $K = \mathbb{C}$ and consider C and $\mathrm{Jac}(C)$ as real manifolds. Then $\theta_P(C)$ gives a class in the Betti cohomology group $H^2(\mathrm{Jac}(C), \mathbb{C})$.

The multiplication by -1 map of $\mathrm{Jac}(C)$ acts as -1 on $H^1(\mathrm{Jac}(C), \mathbb{C})$, and since

$$H^2(\mathrm{Jac}(C), \mathbb{C}) \simeq \wedge^2 H^1(\mathrm{Jac}(C), \mathbb{C}),$$

the involution -1 acts as identity on $H^2(\mathrm{Jac}(C), \mathbb{C})$.

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The Ceresa cycle $\theta_P(C) - [-1] \circ \theta_P(C)$ is homologically trivial, but need not be algebraically trivial.

Theorem (Ceresa,1983)

On a generic Jacobian variety $\mathrm{Jac}(C)$ *of dimension* $g \geq 3$, *the Ceresa cycle is* **not** *algebraically trivial.*

Hyperelliptic curves – definition

Definition

A genus $g \geq 2$ **hyperelliptic curve** is the smooth projective model of an affine curve in $\mathbb{A}^2$ defined by an equation of the following form:

$$C_f : y^2 = f(x)$$

where f is a degree $2g+1$ or $2g+2$ polynomial with distinct roots.

A hyperelliptic curve C_f has an involution given by

$$\iota : (x, y) \mapsto (x, -y)$$

and $C_f \to C_f/\langle\iota\rangle : (x, y) \mapsto x$ is a degree 2 map $C_f \to \mathbb{P}^1$.

The existence of a degree 2 map to $\mathbb{P}^1$ can also be taken as the definition for a curve to be hyperelliptic.

Hyperelliptic curves have algebraically trivial Ceresa cycle

Let $P \in C_f(K)$ with $\iota(P) = P$ and consider the Ceresa cycle $\theta_P - \overline{\theta}_P$.

For any point $Q \in C_f(\overline{K})$, since $Q + \iota(Q) - 2P$ is a principal divisor,

$$[\iota(Q) - P] \equiv [P - Q]$$

meaning the involution on $\mathrm{Jac}(C_f)$ induced by ι is multiplication by -1.

Thus, $\theta_P(C) = \theta_P \circ \iota(C) = [-1] \circ \theta_P(C)$.

This implies the Ceresa cycle is trivial for the pointed curve (C_f, P) and it is algebraically trivial for (C_f, x) for any point $x \in C_f(K)$.

Question (Clemens)

For a genus 3 curve, is its Ceresa cycle algebraically trivial if and only if it is hyperelliptic?

Previous work on Ceresa cycle

Up to algebraic equivalence:

- Fermat curve $F : x^4 + y^4 = 1$ has algebraically nontrivial Ceresa cycle. (B. Harris)
- There exist examples of genus 3 curves whose Ceresa cycle is of infinite order modulo algebraic equivalence. These curves have Jacobians isogenous to E^3 where E is a CM elliptic curve.(J. Top)
- In the tropical analog, a generic tropical curve whose underlining graph has a K4 has algebraically nontrivial Ceresa cycle. (I. Zharkov)

Up to rational equivalence:

- For degree n Fermat curves where n has a prime divisor > 7 has Ceresa cycle of infinite order in $\mathrm{CH}_1(\mathrm{Jac}(C))$.
  (P. Eskandari– V. K. Murty)

The Ceresa class (étale)

Via a cycle class map, the Ceresa cycle has a cycle class

$$\nu(C, P) \in H^1(G_K, \mathbb{L})$$

where $G_K = \mathrm{Gal}(\overline{K}/K)$, $\mathbb{L} = \wedge^3 H(-1)$, $H = H^1_{\text{ét}}(C_{\bar{K}}, \mathbb{Z}_\ell(1))$.

Let $q \in \wedge^2 H(-1)$ representing the polarization and $\tilde{\mathbb{L}} = \mathbb{L}/(H \wedge q)$. From $\nu(C, P)$, there is an induced class

$$\nu(C) \in H^1(G_K, \tilde{\mathbb{L}})$$

which is independent of the choice or existence of a rational base point P and only depends on the rational equivalence class of the Ceresa cycle.

In this talk, we refer to the class $\nu(C)$ **the Ceresa class**.

$$\nu(C) \neq 0 \Rightarrow \theta_P(C) - \overline{\theta}_P(C) \text{ nontrivial in } \mathrm{CH}_1(\mathrm{Jac}(C)) \text{ for any } P.$$

Results – the Fricke-Macbeath curve and its quotient

Theorem (Bisogno–L.–Litt–Srinivasan, 2020)

1. *Let C be a genus 7 curve over a number field K, such that* $\mathrm{Aut}\, C_{\overline{K}} \simeq \mathrm{PSL}_2(8)$. *The Ceresa class $\nu(C)$ is torsion.*
2. *If $\omega \in Aut(C)$ is any element of order 2, then the quotient C/ω is a non-hyperelliptic curve of genus 3 and $\nu(C/\omega)$ is torsion.*

It is known that such curves C (the Fricke-Macbeath curves) exist. They are Hurwitz curves of genus 7, i.e. $\#\,\mathrm{Aut}\, C_{\overline{K}} = 84(g-1)$.
One such curve has an affine model over $\mathbb{Q}$ given by Bradley Brock:

$$1 + 7xy + 21x^2y^2 + 35x^3y^3 + 28x^4y^4 + 2x^7 + 2y^7 = 0.$$

A model of C/ω for some ω is given by Jaap Top and Carlo Verschoor:

$$5x^4 + 12x^3y + 6x^2y^2 - 4xy^3 + 4y^4 - 28x^3z + 16x^2yz - 24xy^2z + 16y^3z + 24x^2z^2 - 10y^2z^2 - 12xz^3 + 8yz^3 + 3z^4 = 0$$

Results – semistable curves over $\mathbb{C}((t))$

Theorem (Corey–Ellenberg–L., in progress)

Let C be a smooth, projective curve over $\mathbb{C}((t))$ with semistable reduction at the special fiber. The Ceresa class $\nu(C)$ is torsion.

Remark

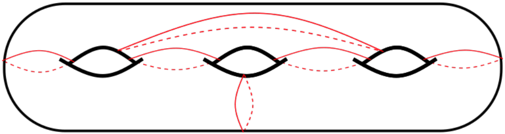
We give an algorithm which explicitly computes the Ceresa class

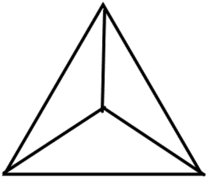
$$\nu(C) \in H^1(\langle\rho\rangle, \tilde{\mathbb{L}}) \simeq \tilde{\mathbb{L}}/(\rho - I)\tilde{\mathbb{L}}$$

where ρ denotes the topological generator of $G_{\mathbb{C}((t))} \simeq \hat{\mathbb{Z}}$ and $\tilde{\mathbb{L}} = \mathbb{L}/H \wedge q$ using only the degeneration type of the special fiber.

Key: For a curve over $\mathbb{C}((t))$ with semistable reduction, the Galois action on its fundamental group can be read off from the degeneration data of its special fiber. Next we explain how this data relates to the Ceresa class.

Example–The complete graph with 4 vertices





The picture above is showing a genus 3 Riemann surface with 6 disjoint closed curves. Consider a curve C over $\mathbb{C}((t))$ whose monodromy is the product of the Dehn twists around the 6 red loops each with multiplicity 1. Take an integral model $\tilde{C}/\mathbb{C}[\![t]\!]$ of C. The special fiber of $\tilde{C}$ is a stable curve whose dual graph is given by 4 mutually meeting lines.

Theorem (Corey–Ellenberg–L., in progress)

The Ceresa class of C is non-trivial and is of order 16.

Galois action on the geometric fundamental group

Let π be the geometric pro-ℓ fundamental group of C with base point $P \in C(K)$ and consider the lower central series of $\pi = \pi_{1,\text{ét}}^{\ell}(C_{\bar{K}}, P)$:

$$\pi = L^1\pi \supset L^2\pi \supset L^3\pi \supset \dots, \text{ where } L^{k+1}\pi = [\pi, L^k\pi].$$

Since $P \in C(K)$, it induces a splitting of the sequence

$$1 \to \pi_{1,\text{ét}}^{\ell}(C_{\bar{K}}, P) \to \pi_{1,\text{ét}}^{\ell}(C, P) \to G_K \to 1.$$

We get a G_K action on π. Furthermore, we get the following sequence of G_K-modules where $L^2\pi/L^3\pi, \pi/L^2\pi$ are abelian:

$$1 \to L^2\pi/L^3\pi \to L^1\pi/L^3\pi \to L^1\pi/L^2\pi \to 1.$$

By the work of Hain–Matsumoto, twice of the class in the cohomology group $H^1(G_K, \mathrm{Hom}(L^1\pi/L^2\pi, L^2\pi/L^3\pi))$ which characterizes the G_K action on $L^1\pi/L^3\pi$ is equal to the Ceresa class $\nu(C, P)$.

Group theoretic Ceresa class and base-point independence

Let ℓ be a prime and G a finitely generated pro-ℓ group with torsion-free abelianization G^{ab}. Define the completed ℓ-adic group ring of G as

$$\mathbb{Z}_\ell[[G]] := \varprojlim_{G \twoheadrightarrow H} \mathbb{Z}_\ell[H].$$

Here the inverse limit is taken over all finite groups H which are continuous quotients of G. Let $I \subset \mathbb{Z}_\ell[[G]]$ be the augmentation ideal.

In BLLS, we construct group cohomology classes

$$\mathrm{MD}_G \in H^1(\mathrm{Aut}(G), \mathrm{Hom}(I/I^2, I^2/I^3))$$

$$J_G \in H^1(\mathrm{Out}(G), \tilde{\mathbb{L}}_G)$$

where $\tilde{\mathbb{L}}_G$ is a quotient of $\mathrm{Hom}(I/I^2, I^2/I^3))$ such that when we take $G = \pi = \pi_{1,\text{ét}}^{\ell}(C_{\bar{K}}, P)$, we have $2\mathrm{MD}_\pi = \nu(C, P)$ and $2J_\pi = \nu(C)$.

Properties – Stability under base change

Proposition (Bisogno–L.–Litt–Srinivasan, 2020)

Let K be a field and X a smooth, geometrically connected curve over K. Let ℓ be a prime and $\nu(X)$ the Ceresa class. Let L/K be a finite extension. Then $\nu(X_L)$ is torsion if and only if $\nu(X)$ is torsion.

Proof: Consider the inclusion of Galois groups $G_L \hookrightarrow G_K$ which induces

$$\mathrm{Res}(\nu(X_K)) = \nu(X_L),$$
$$\text{co-Res} \circ \mathrm{Res}(\nu(X_K)) = [L:K]\nu(X_K).$$

Upshot: Given a curve X/K, We can work over a field extension L/K over which all the automorphisms of $X_{\overline{K}}$ is defined and conclude for $\nu(X_K)$.

Properties – Aut(X)-invariance

Proposition (Bisogno–L.–Litt–Srinivasan, 2020)

Let $B \subset \mathrm{Aut}_K(X)$ be a finite subgroup such that

$$H^0(B, \tilde{\mathbb{L}}_G) = \tilde{\mathbb{L}}_G^B = 0.$$

Then the Ceresa class $\nu(X)$ is torsion with order $d \mid \#B$.

Proof: Apply the inflation-restriction sequence to the group extension

$$1 \to B \to G_K \times B \to G_K \to 1,$$

which gives

$$0 \to H^1(G_K, \tilde{\mathbb{L}}_G^B) \to H^1(G_K \times B, \tilde{\mathbb{L}}_G) \to H^1(B, \tilde{\mathbb{L}}_G)^{G_K} \to \cdots$$

Since B is a finite group, for any finitely generated B-module M, elements of $H^i(B, M)$ has order $d \mid \#B$.

Properties – respect dominant maps between curves

Theorem (Bisogno–L.–Litt–Srinivasan, 2020)

Let X be a curve over a number field K, and let $f : X \to Y$ be a dominant map of curves over K.

If $\nu(X)$ is torsion, then $\nu(Y)$ is torsion.

Application: we conclude the smooth quartic in $\mathbb{P}^2$ defined by the following equation has torsion Ceresa class.

$$5x^4 + 12x^3y + 6x^2y^2 - 4xy^3 + 4y^4 - 28x^3z + 16x^2yz - 24xy^2z + 16y^3z + 24x^2z^2 - 10y^2z^2 - 12xz^3 + 8yz^3 + 3z^4 = 0$$

Thanks