Singular moduli for real quadratic fields

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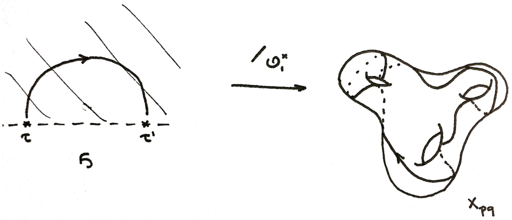

Joint work with Henri Darmon, Alice Pozzi, Yingkun Li

Outline

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Let us begin with the observation, often attributed to Ramanujan, that

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots$$

Why is this so close to an integer? Answer comes from the theory of *complex multiplication*, by looking at the j-function

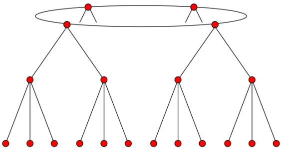
$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \qquad q = e^{2\pi iz}$$

This function satisfies

$$j\left(\frac{az+b}{cz+d}\right) = j(z), \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

The values of this function at $z \in K$ quadratic imaginary are called *singular moduli*. They are always algebraic integers, e.g

$$\begin{aligned} j(\sqrt{-1}) &= 1728 \\ j(\sqrt{-5}) &= 2^6 \cdot 5 \cdot (884\sqrt{5} + 1975) \\ j(\sqrt{-14}) &= 2^3\left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3 \end{aligned}$$



Let us explore the isogeny volcano (Cfr. Pete Clark/Drew Sutherland)

- $j(\sqrt{-5})$ generates $\mathbf{Q}(\sqrt{5})$,
- $j(2\sqrt{-5})$ generates $\mathbf{Q}(\sqrt{\frac{1+\sqrt{5}}{2}})$,
- ...

Theorem (Complex multiplication)

All finite abelian extensions of K are (essentially) generated by

$$j(z) \;\; z \in K, \qquad \exp(\pi i z) \;\; z \in \mathbf{Q}\,.$$

Understand the $K(j(\tau))$ (= ring class fields) and the Galois action on the set of $j(\tau)$. Has many applications, e.g. proof of Euler's conjecture:

$$p = x^2 + 27y^2 \iff \begin{cases} p \equiv 1 \pmod 3 \text{ and} \\ t^3 - 2 \in \mathbf{F}_p[t] \text{ has a root.} \end{cases}$$

A singular modulus is an integer if and only if argument generates an order of class number one. There is a finite list! The maximal ones are:

Field	$E_{\mathbf{Q}}$ with CM by maximal order	$j(E)$
$\mathbf{Q}(\sqrt{-1})$	$y^2 = x^3 + x$	$2^6 \cdot 3^3$
$\mathbf{Q}(\sqrt{-2})$	$y^2 = x^3 + x$	$2^6 \cdot 5^3$
$\mathbf{Q}(\sqrt{-3})$	$y^2 + xy = x^3 - x^2 - 2x - 1$	0
$\mathbf{Q}(\sqrt{-7})$	$y^2 = x^3 + 4x^2 + 2x$	$-3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-11})$	$y^2 + y = x^3 - x^2 - 7x + 10$	-2^{15}
$\mathbf{Q}(\sqrt{-19})$	$y^2 + y = x^3 - 38x + 90$	$-2^{15} \cdot 3^3$
$\mathbf{Q}(\sqrt{-43})$	$y^2 + y = x^3 - 860x + 9707$	$-2^{18} \cdot 3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-67})$	$y^2 + y = x^3 - 7370x + 243528$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$
$\mathbf{Q}(\sqrt{-163})$	$y^2 + y = x^3 - 2174420x + 1234136692$	$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$

This explains the observation on our first slide!

$$-262537412640768000 = j\left(\frac{1+\sqrt{-163}}{2}\right) = -e^{\pi\sqrt{163}} + 744 + (\text{very small}).$$

CM theory was believed to have reached satisfactory conclusion in early 20th century. Until Gross–Zagier got their hands on it! Observe

$$\begin{aligned} j\left(\frac{1+\sqrt{-67}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) &= -2^{15}\cdot 3^3\cdot 5^3\cdot 11^3 + 2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3 \\ &= 2^{15}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 13\cdot 139\cdot 331 \end{aligned}$$

Differences are very smooth! But wait... doesn't ABC say this must be rare? Luckily, there are only finitely many class number one orders!

Remark 1. This is the *class number one* problem, solved by Heegner by finding all integral points on $X_{\mathrm{ns}}^+(24)$. Amusing: Can also use $X_{\mathrm{ns}}^+(13)$, solved in Balakrishnan–Dogra–Müller–Tuitman–V. using p-adic heights.

Remark 2. Note that according to Gauß, *real* quadratic fields $K/\mathbf{Q}$ should have class number one very often! Keep that in mind in what follows.

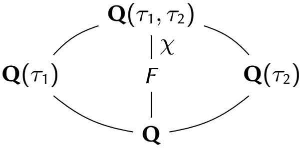
Let τ_1, τ_2 be two CM points in $\mathcal{H}_\infty = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.
Gross and Zagier (1985) find explicit formula for

$$\mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right) \quad \in \mathbf{Z}$$

- **Algebraic proof:** Uses CM elliptic curves! Its q-adic valuation is given in terms of arithmetic intersection of embeddings

$$\mathbf{Q}(\tau_1), \mathbf{Q}(\tau_2) \ \hookrightarrow \ B_{\infty q} = \text{Quat alg conductor } \infty q$$

- **Analytic proof:** Fourier coefficients of Hecke's real analytic Eisenstein series over F, attached to the character χ:



Real analytic Hilbert Eisenstein series $E_s(z_1, z_2)$ defined by Hecke:

$$\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\Delta_1\Delta_2)} \chi(\mathfrak{a})\,\mathrm{Nm}(\mathfrak{a})^{1+2s} \sideset{}{'}\sum_{(m,n)\in\mathfrak{a}^2/U} \frac{y_1^s y_2^s}{(mz_1+n)(m'z_2+n')|mz_1+n|^{2s}|m'z_2+n'|^{2s}}$$

Gross–Zagier consider its diagonal restriction $E_s(z, z)$ and show

- When $s = 0$, have $E_s(z, z) = 0$,
- The holomorphic projection of the first derivative

$$\left(\frac{\partial}{\partial s} E_s(z, z)\right)\Bigg|_{s=0}^{\mathrm{hol}}$$

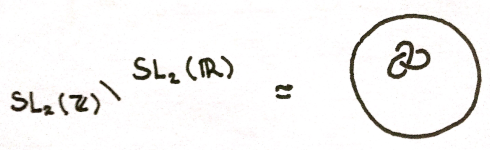
  has Fourier coefficients related to $\log \mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right)$.
- The holomorphic projection must vanish!
  $\Rightarrow$ formula for $\mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right)$.

Remark: Does not use CM elliptic curves!

Outline

The work of Duke–Imamoḡlu–Tóth

Inspiration comes from work of Duke–Imamoḡlu–Tóth on linking numbers of modular geodesics.



If $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ is hyperbolic, get associated knot

$$\begin{aligned} \mathrm{Kn}(\gamma) &\hookrightarrow \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R}) \\ t &\mapsto \mathrm{SL}_2(\mathbf{Z}) g \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}, \quad \text{where } g^{-1}\gamma g = \text{diagonal} \end{aligned}$$

- Linking $\mathrm{Kn}(\gamma)$ and trefoil $\leftrightarrow$ Dedekind–Rademacher cocycle (Ghys)
- Linking $\mathrm{Kn}(\gamma_1)$ and $\mathrm{Kn}(\gamma_2)$ $\leftrightarrow$ Knopp cocycle (DIT)

I. The Dedekind–Rademacher cocycle

Consider E_2, the Eisenstein series of weight 2, defined by

$$\frac{\pi i}{6} E_2(z) = \sum_{m \in \mathbf{Z}} \sum_{n \in \mathbf{Z}}{}' \frac{1}{(mz+n)^2}.$$

It is *nearly* invariant under $\mathrm{SL}_2(\mathbf{Z})$, in the sense that

$$(cz+d)^{-2} E_2\left(\frac{az+b}{cz+d}\right) = E_2(z) - \frac{12c}{cz+d}$$

The abstract map $\gamma \mapsto 12c/(cz+d)$ is a weight 2 *cocycle* for $\mathrm{SL}_2(\mathbf{Z})$:

$$f : \mathrm{SL}_2(\mathbf{Z}) \to \mathcal{O}_2, \qquad f(\gamma_1\gamma_2) = f(\gamma_1)^{\gamma_2} f(\gamma_2)$$

These are classified up to equivalence by $\mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}_2)$.

I. The Dedekind–Rademacher cocycle

It lifts uniquely to a weight 0 cocycle:

$$0 \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}) \xrightarrow{d} \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}_2) \longrightarrow 0$$

The Dedekind–Rademacher symbol $\Phi(\gamma) \in \mathbf{Z}$ is defined by

$$\log \Delta(\gamma z) - \log \Delta(z) = 6 \log(-(cz+d)^2) + 2\pi i \Phi(\gamma).$$

since $\operatorname{dlog} \Delta(z) = E_2(z)$, the unique lift is given by the right hand side!
When applied to a hyperbolic matrix γ with fixed point τ, we get

$$6 \log(-(cz+d)^2) \qquad + \quad 2\pi i \Phi(\gamma)$$

$$\Big\downarrow z=\tau \qquad\qquad \Big\downarrow z = i\infty \quad \text{(+ homog.)}$$

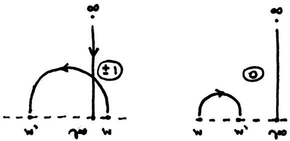
$$12 \log(\text{fundamental unit}) \qquad \mathit{Link}(\mathrm{Kn}(\gamma), \text{trefoil})$$

II. The Knopp cocycle

The *Knopp cocycle* in $Z^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}_2)$ attached to an RM point τ is

$$\gamma \longmapsto \sum_{w \in \mathrm{SL}_2(\mathbf{Z})\tau} \frac{\{\infty \to \gamma\infty\} \cap \{w \to w'\}}{z - w}$$

where the exponent is the intersection number of the geodesic from w' to w with the geodesic from ∞ to $\gamma\infty$, and hence ± 1 or 0.



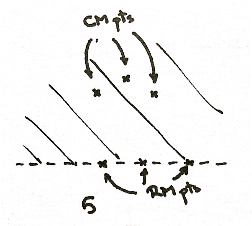
In similar way, Duke–Imamoḡlu–Tóth extract linking $\mathrm{Kn}(\gamma_1)$ and $\mathrm{Kn}(\gamma_2)$.

Remark. An analogue of $E_2(z)$ for the Knopp cocycle was constructed in a different article of Duke–Imamoglu–Toth (2011). It's a deep object.

Outline

∞ versus p

Cannot evaluate $j(z)$ at arguments in a real quadratic field K.



Naive issue: ∞ is split in K.
Naive solution: Plenty of finite primes p are not split in K.

points?	$\mathcal{H}_\infty$	$\mathcal{H}_2$	$\mathcal{H}_3$	$\mathcal{H}_5$	$\mathcal{H}_7$	$\mathcal{H}_{11}$
$\mathbf{Q}(i)$	yes	yes	yes	no	yes	yes
$\mathbf{Q}(\sqrt{5})$	no	yes	yes	yes	yes	no

Work over p-adic numbers (cfr. Liang Xiao's course), where p is inert in K.

With Henri Darmon, we upgrade linking number ideas to setting

$$\begin{aligned} \Gamma &= \mathrm{SL}_2(\mathbf{Z}[1/p]) \\ \mathcal{M} &= \text{Meromorphic functions on } \mathcal{H}_p = \mathbf{P}^1(\mathbf{C}_p) \setminus \mathbf{P}^1(\mathbf{Q}_p) \end{aligned}$$

The dichotomy Dedekind–Rademacher / Knopp cocycle becomes:

(Darmon–Dasgupta 2006) Constructed p-adic invariants

$$J_{\mathrm{DR}}(\tau) \in \mathbf{C}_p \, .$$

Give p-units in ring class field of τ. *E.g.* $p = 7$ and $\tau = \frac{-17+\sqrt{321}}{4}$ gives

$$7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 120922465776x + 7^{16}$$

Independent proofs by Darmon–Pozzi–V. /Dasgupta–Kakde (forthcoming).
Also purely archimedean variant by Charollois–Darmon, no proof.

(Darmon–V. 2020) Constructed p-adic invariants

$$J_p(\tau_1, \tau_2) \in \mathbf{C}_p$$

for pair of RM points τ_1, τ_2 which appear to be good analogues of the quantity $J_\infty(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2)$ appearing in Gross–Zagier.

Let $\Delta_1 = 13$, then for below choices of p and τ consider the quantity

$$J_p\left(\frac{1+\sqrt{13}}{2},\ \tau\right).$$

Can compute these numerically (this is not a proof!) and seem to get:

τ	$p = 11$	$p = 19$	$p = 59$
$2\sqrt{2}$	$\frac{3-4\sqrt{-1}}{5}$	$\frac{3-4\sqrt{-1}}{5}$	1
$3\sqrt{2}$	$\frac{11+21\sqrt{-3}}{2\cdot 19}$	$\frac{5-4\sqrt{-6}}{11}$	1
$4\sqrt{2}$	$\frac{57-176\sqrt{-1}}{5\cdot 37}$	$\frac{5-12\sqrt{-1}}{13}$	$\frac{3+4\sqrt{-1}}{5}$
$7\sqrt{2}$	$\frac{118393-8328\sqrt{-14}}{5^2\cdot 59\cdot 83}$	$\frac{93+95\sqrt{-7}}{2^2\cdot 67}$	$\frac{37+9\sqrt{-7}}{2^2\cdot 11}$
$8\sqrt{2}$	$\frac{1312-1425\sqrt{-1}}{13\cdot 149}$	$\frac{43+924\sqrt{-1}}{5^2\cdot 37}$	$\frac{3+4\sqrt{-1}}{5}$
$9\sqrt{2}$	$\frac{11387+12320\sqrt{-3}}{19^2\cdot 67}$	$\frac{43+4100\sqrt{-6}}{11^2\cdot 83}$	1

Observe that for any pair of primes p, q there seems to be some relation

$$\text{“ord}_p\text{”}(q\text{-adic invariant}) = \text{“ord}_q\text{”}(p\text{-adic invariant}).$$

(Gross–Zagier) Let τ_1, τ_2 be CM points, consider

$$J_\infty(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2) \;\in\; \overline{\mathbf{Q}}$$

- Related to real analytic Eisenstein family.
- $\mathrm{ord}_q\, J_\infty(\tau_1, \tau_2) =$ Intersection multiplicities

$$\mathbf{Q}(\tau_1), \mathbf{Q}(\tau_2) \;\hookrightarrow\; B_{\infty q}.$$

(Darmon–V.) Let τ_1, τ_2 be RM points, construct

$$J_p(\tau_1, \tau_2) \;\overset{?}{\in}\; \overline{\mathbf{Q}}$$

- Related to p-adic analytic families.
- $\mathrm{ord}_q\, J_p(\tau_1, \tau_2) \overset{?}{=}$ Intersection multiplicities

$$\mathbf{Q}(\tau_1), \mathbf{Q}(\tau_2) \;\hookrightarrow\; B_{pq}.$$

Towards a proof?

Can relate real quadratic singular moduli to derivatives of p-adic families of modular forms. Then have big (and exclusively p-adic) advantage:

p-Adic families of modular forms

$\updownarrow$

Deformations of Galois representations

(Cfr. Jeremy Booher / David Savitt)

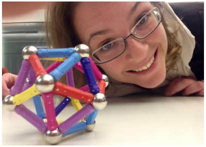
- **(With Darmon and Pozzi)** Proof that $\Theta_{\mathrm{DR}}[\tau] \in \mathcal{O}_H[1/p]^\times$. Use deformation of Eisenstein series in weight $(1, 1)$.
- **(With Darmon and Li)** Proof that p-adic family through a certain modular form of weight $3/2$ is closely related to

$$J_p(\tau_1, D) = \prod_{\mathrm{disc}(\tau_2)=D} J_p(\tau_1, \tau_2)$$

Implies certain algebraicity results (in progress).

And finally...

... a **huge** thanks to the organisers: Jennifer Balakrishnan, Keith Conrad, Álvaro Lozano-Robledo, Christelle Vincent

as well as the lovely people at UConn

for a fantastic CTNT! Let us give them a huge round of applause!!