Isolated points on curves

Bianca Viray
University of Washington

I acknowledge that I live and work on the traditional territories of the Duwamish and Coast Salish peoples.

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BELOV

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Faltings's Theorem ('83)

Let C be a nice algebraic curve over $\mathbb{Q}$.
The curve C has infinitely many $\mathbb{Q}$–points
only if genus$(C) \leq 1$.

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What about $\{P \in C : [\mathbf{k}(P) : \mathbb{Q}] \leq d\}$??

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Let's consider double covers of $\mathbb{P}^1$.

$$y^2 = x(x-1)(x-2)(x-3)(x-4)(x-5)\cdots(x-2g-1)$$

genus $g \geq 2$

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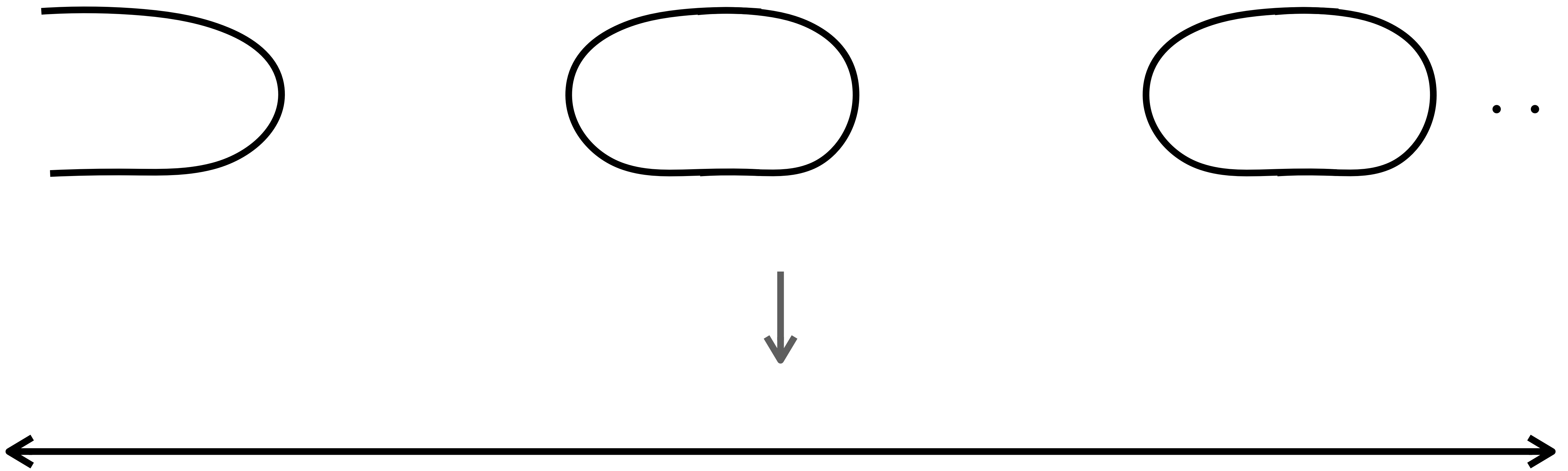

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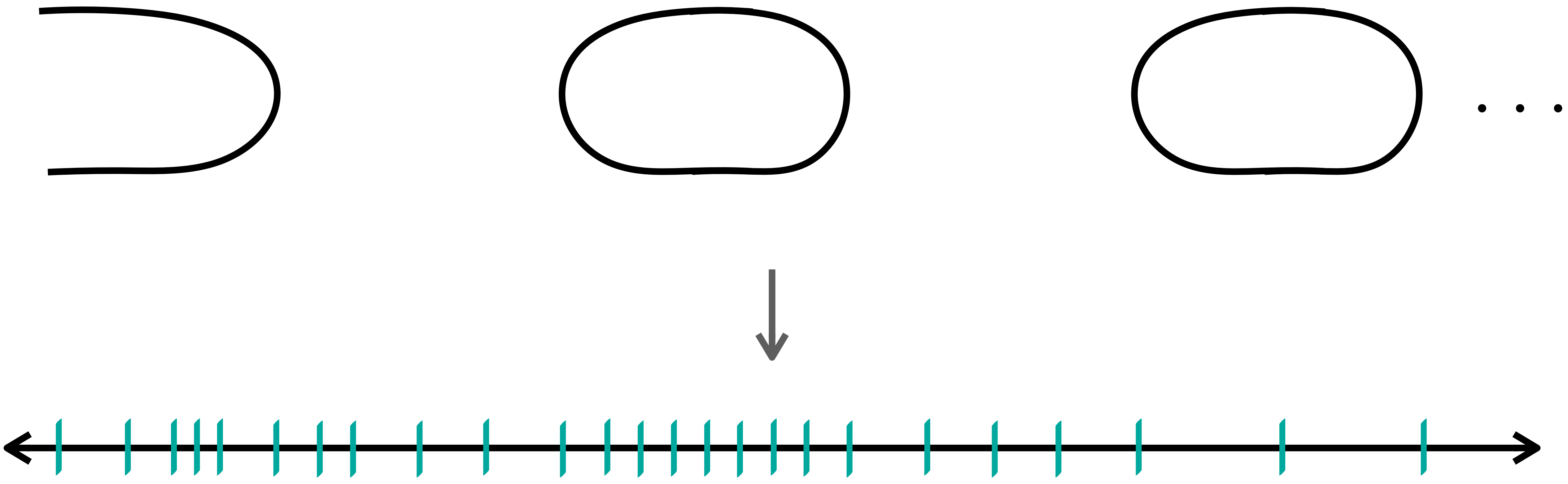

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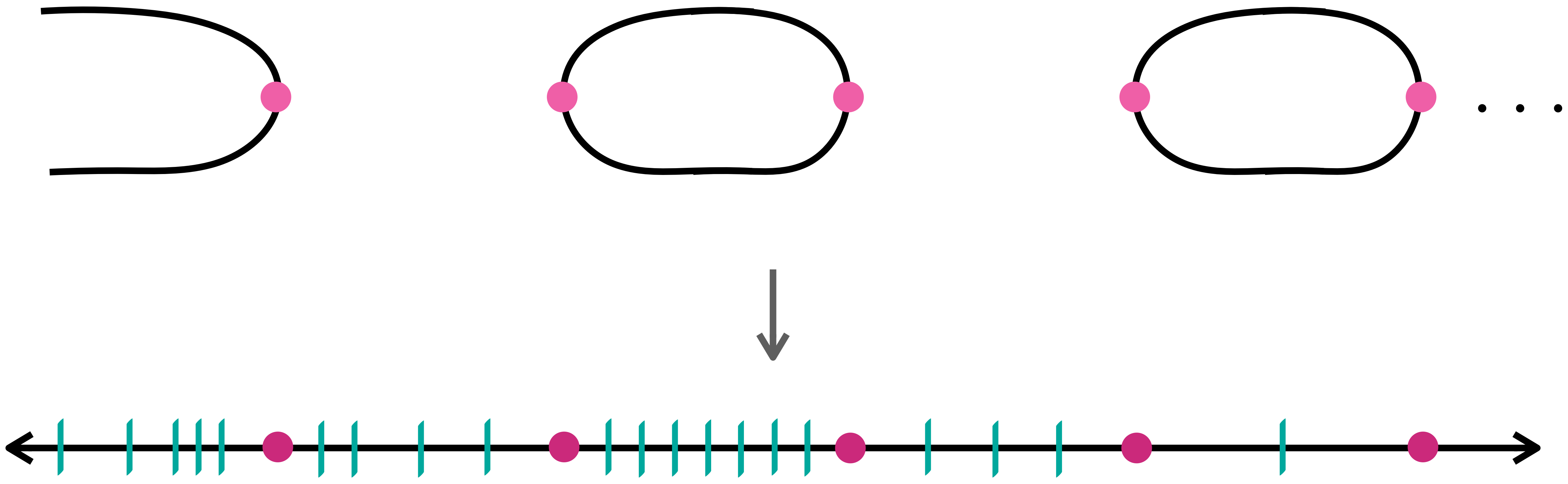

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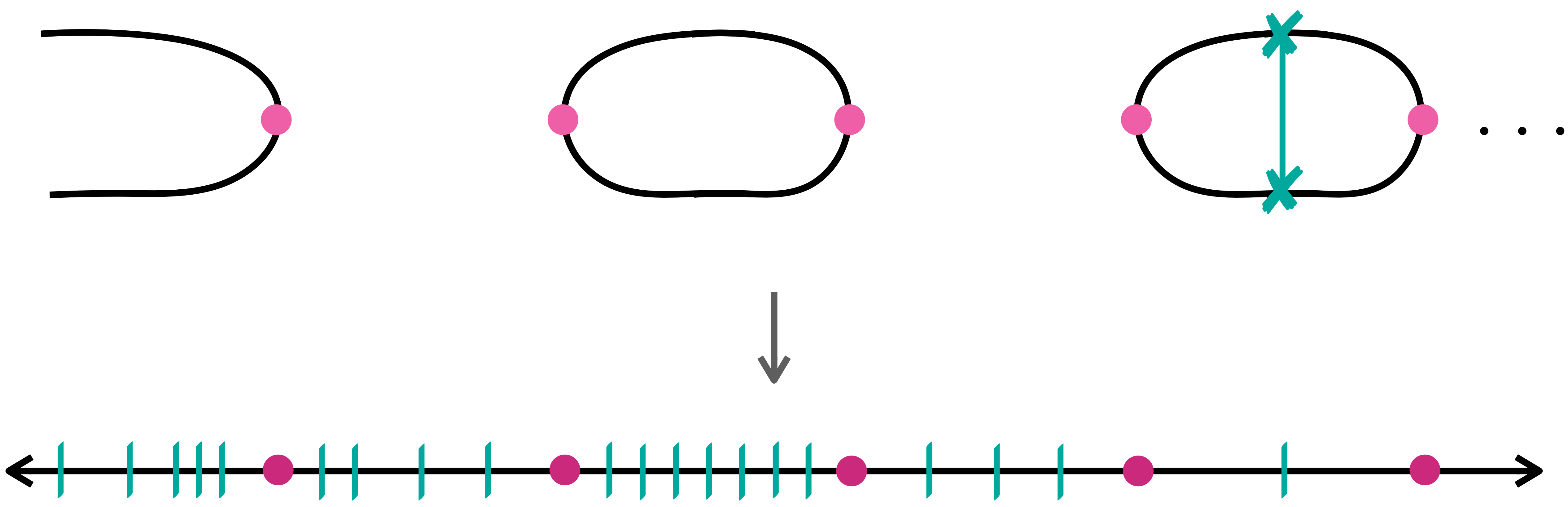

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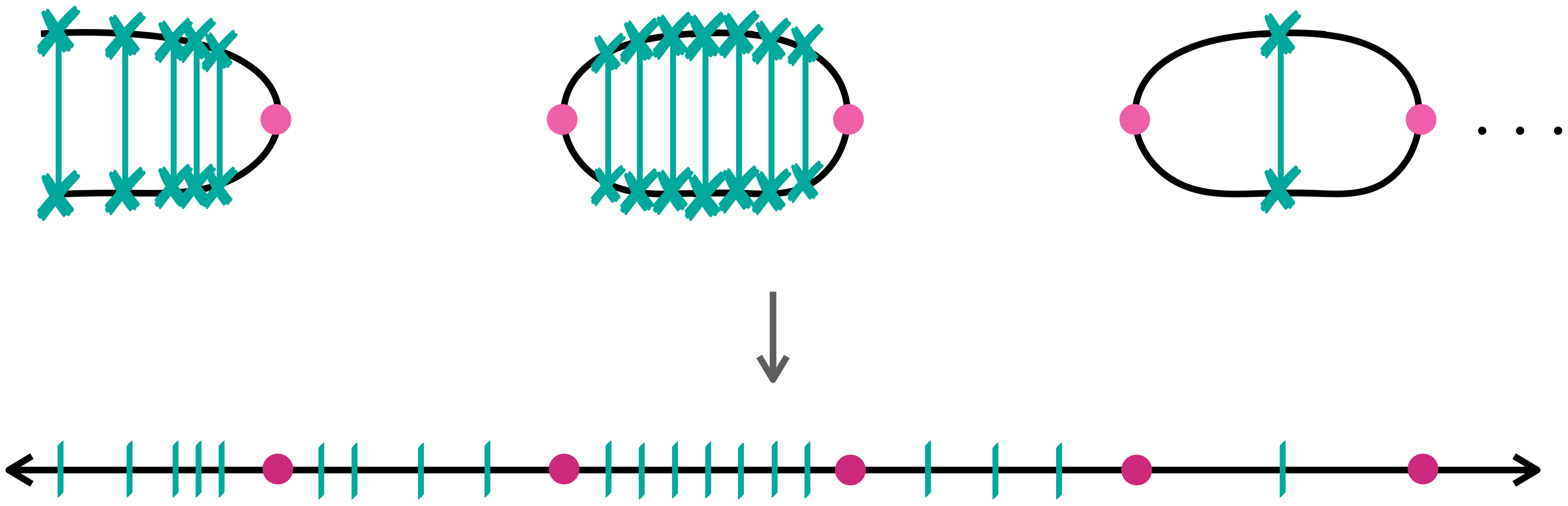

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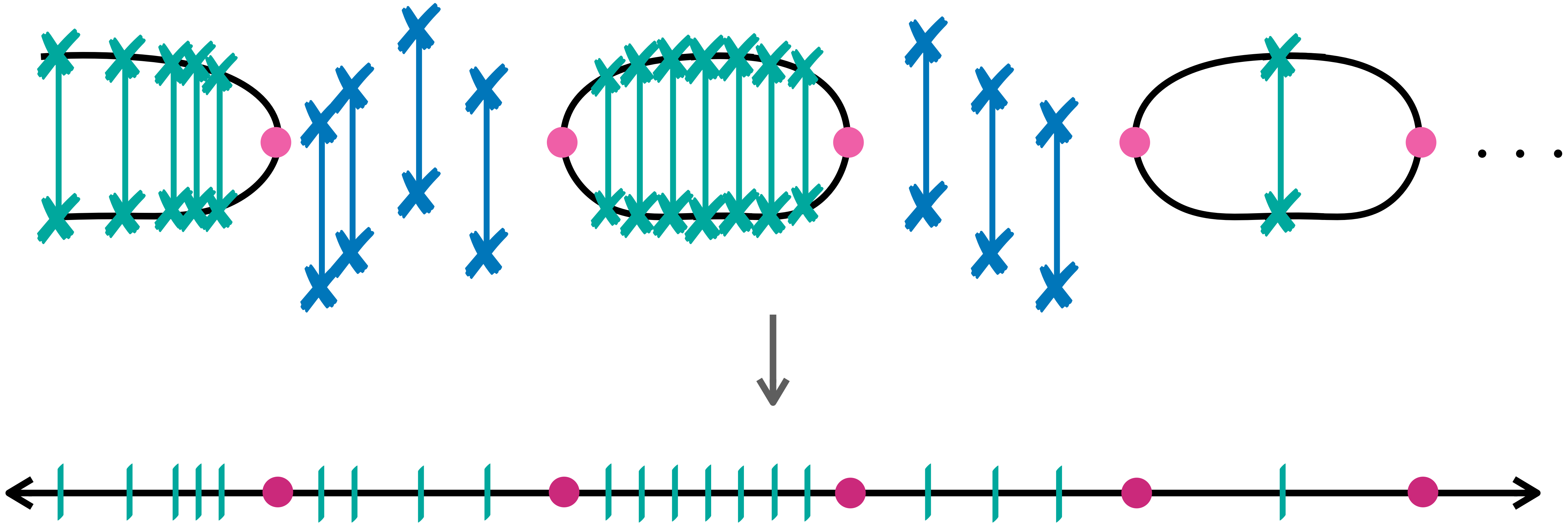

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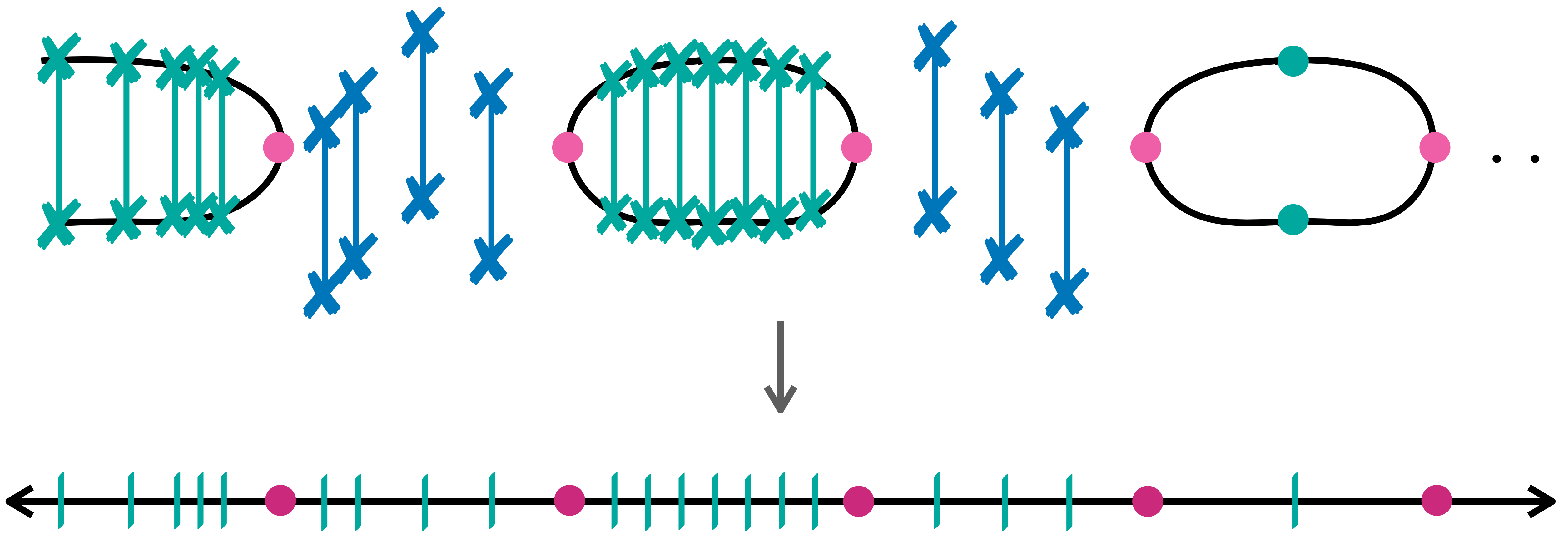

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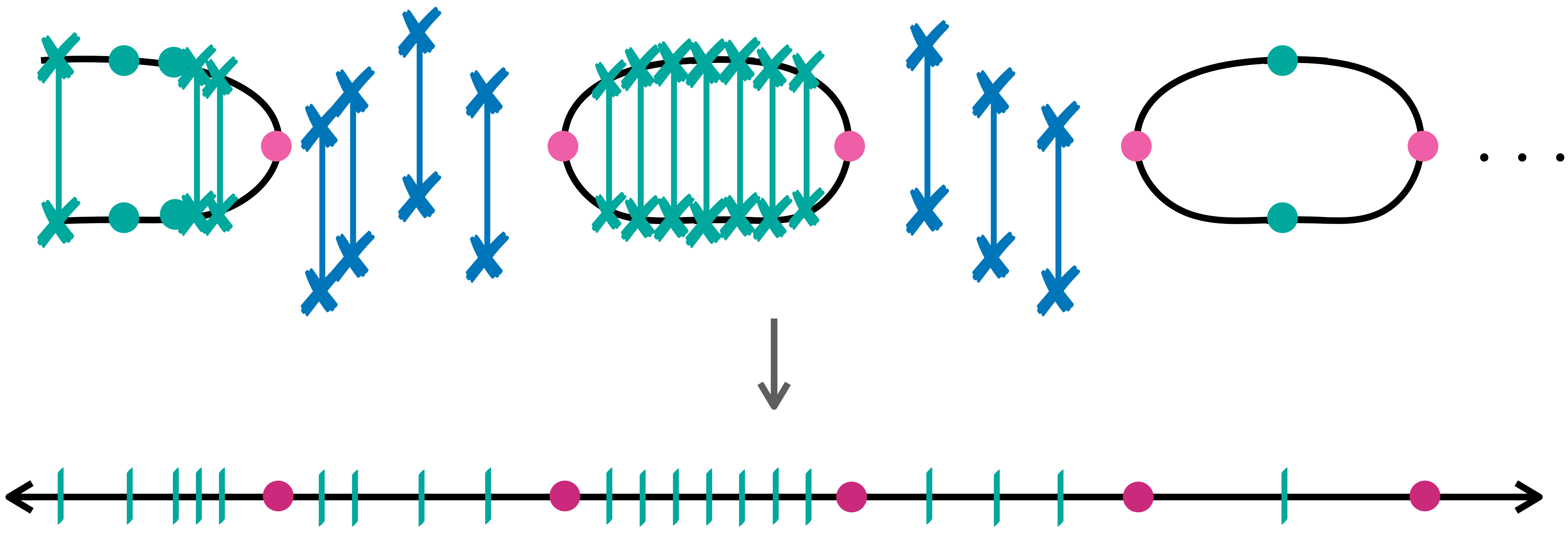
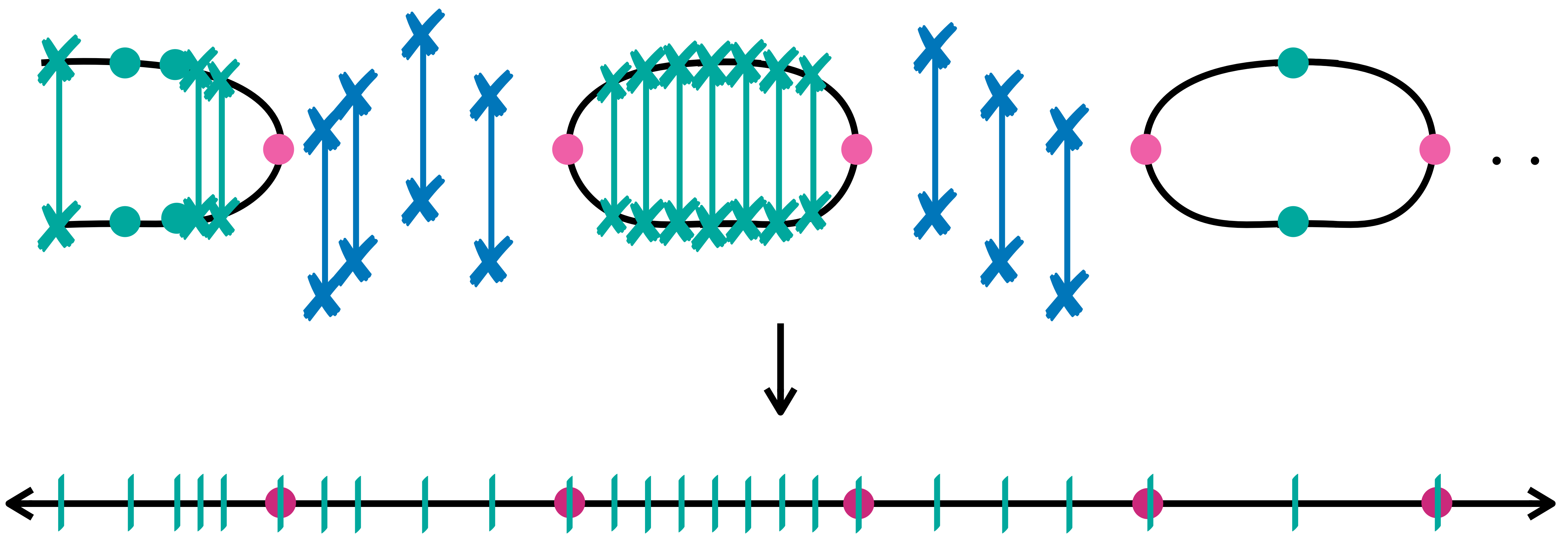


Definition*: Let C be a nice algebraic curve over $\mathbb{Q}$.

A degree d point $x \in C(\overline{\mathbb{Q}})$ is $\mathbb{P}^1$-parametrized if there is a degree d map $f\colon C \to \mathbb{P}^1$ such that $f(x) \in \mathbb{P}^1(\mathbb{Q})$.

Otherwise, x is $\mathbb{P}^1$-isolated.



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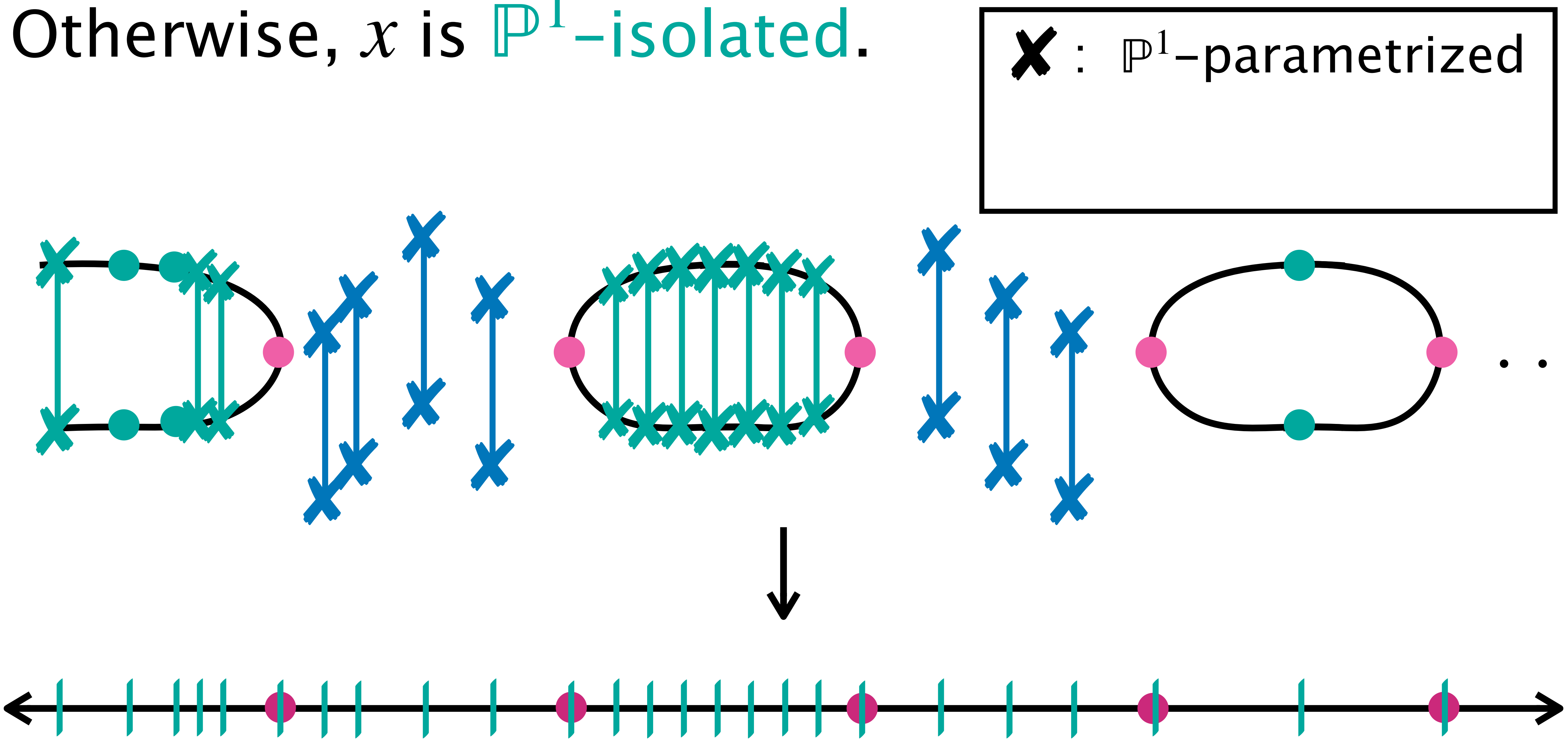

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✗ : $\mathbb{P}^1$–parametrized

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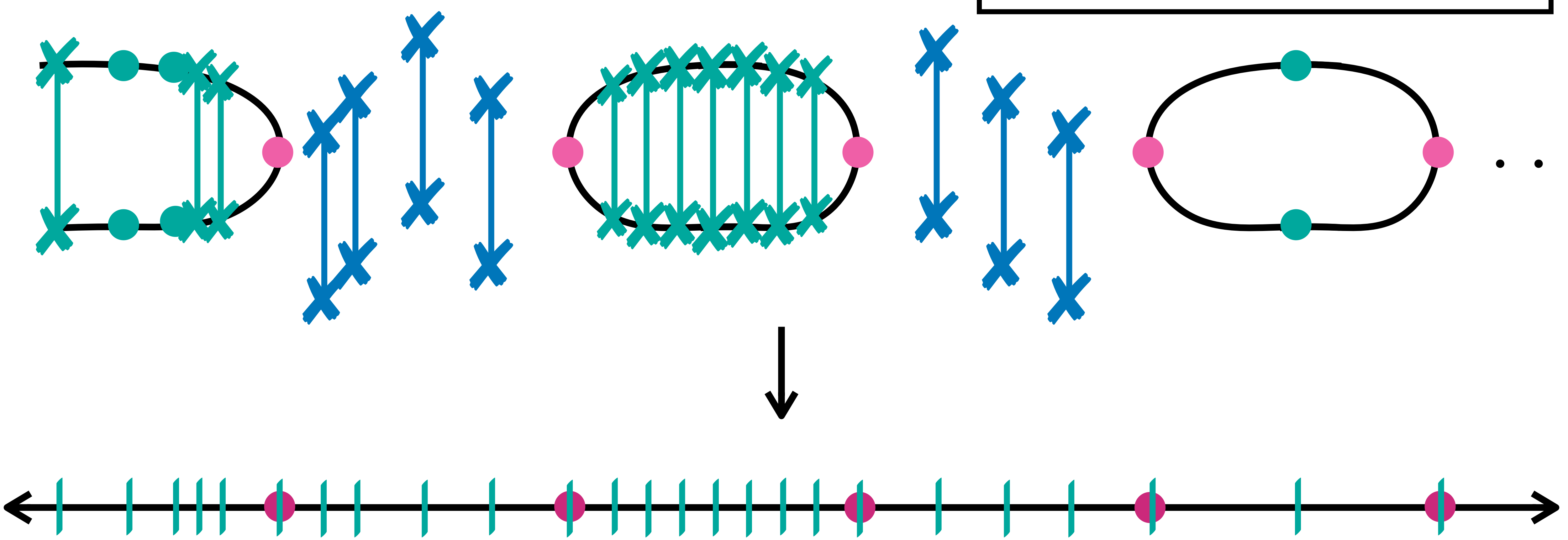
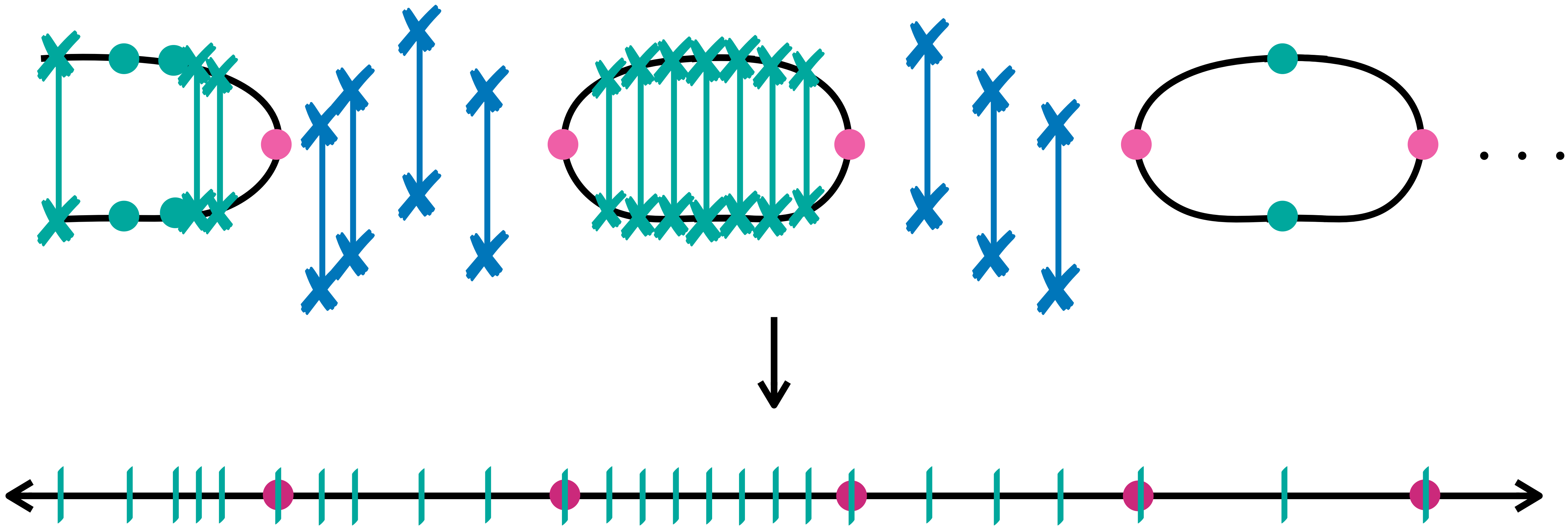
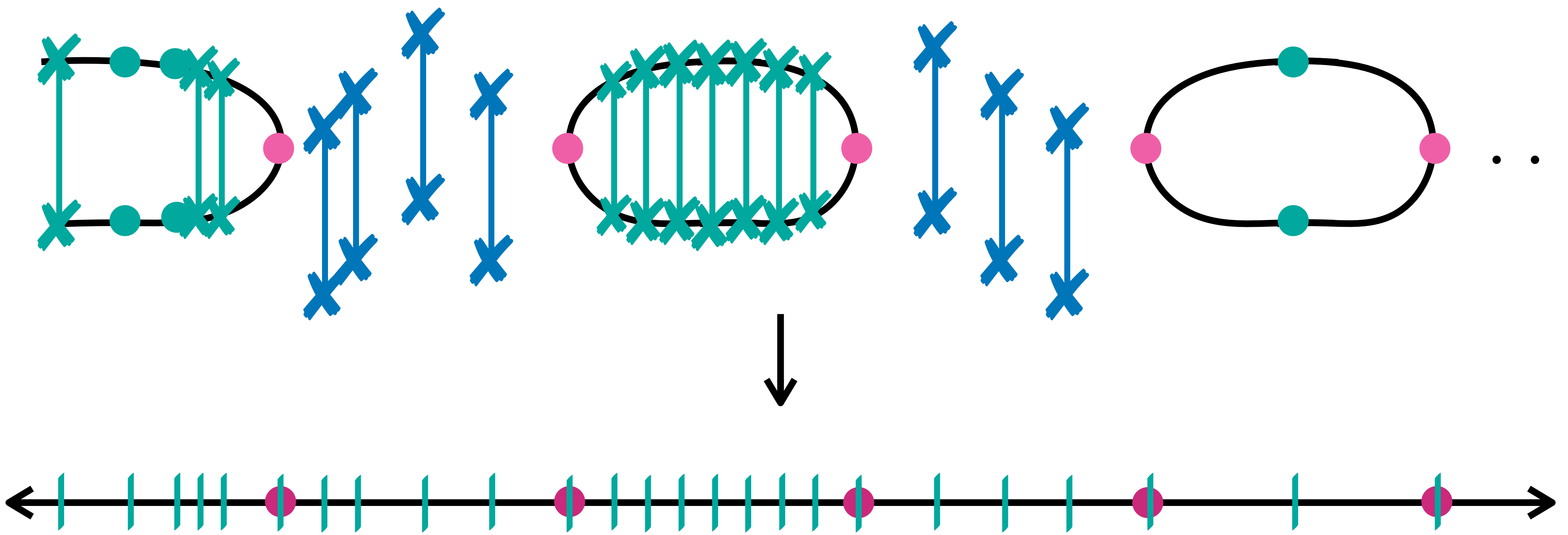


Hilbert Irreducibility Theorem:

A degree d map $f\colon C \to \mathbb{P}^1$ gives infinitely many degree d, $\mathbb{P}^1$-parametrized points.



Are maps to $\mathbb{P}^1$ the only way we get infinitely many degree d points?



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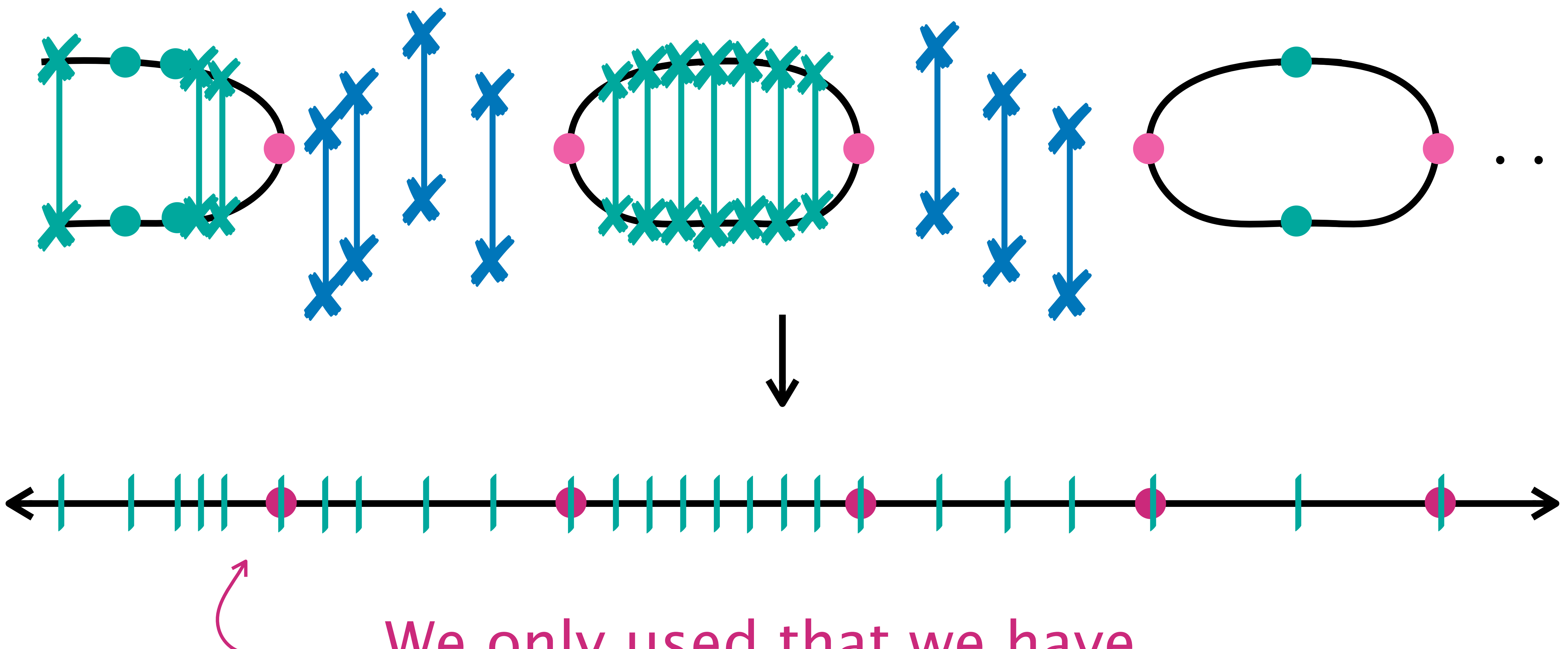

We only used that we have infinitely many rational points!

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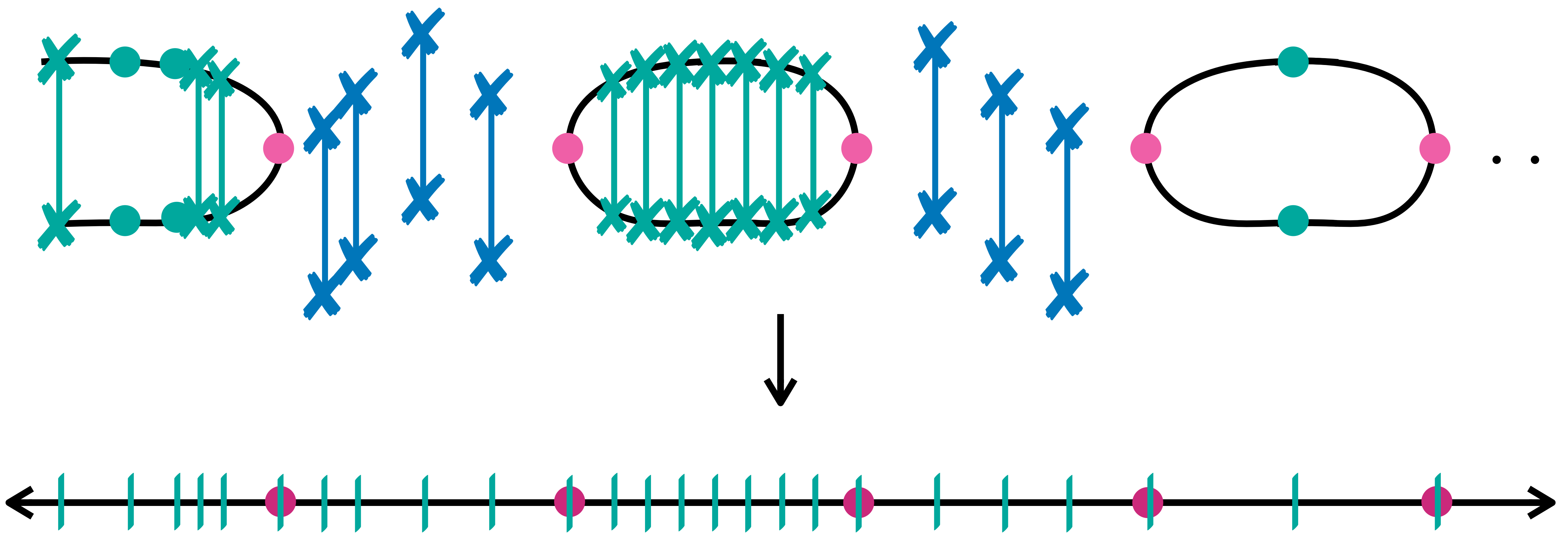
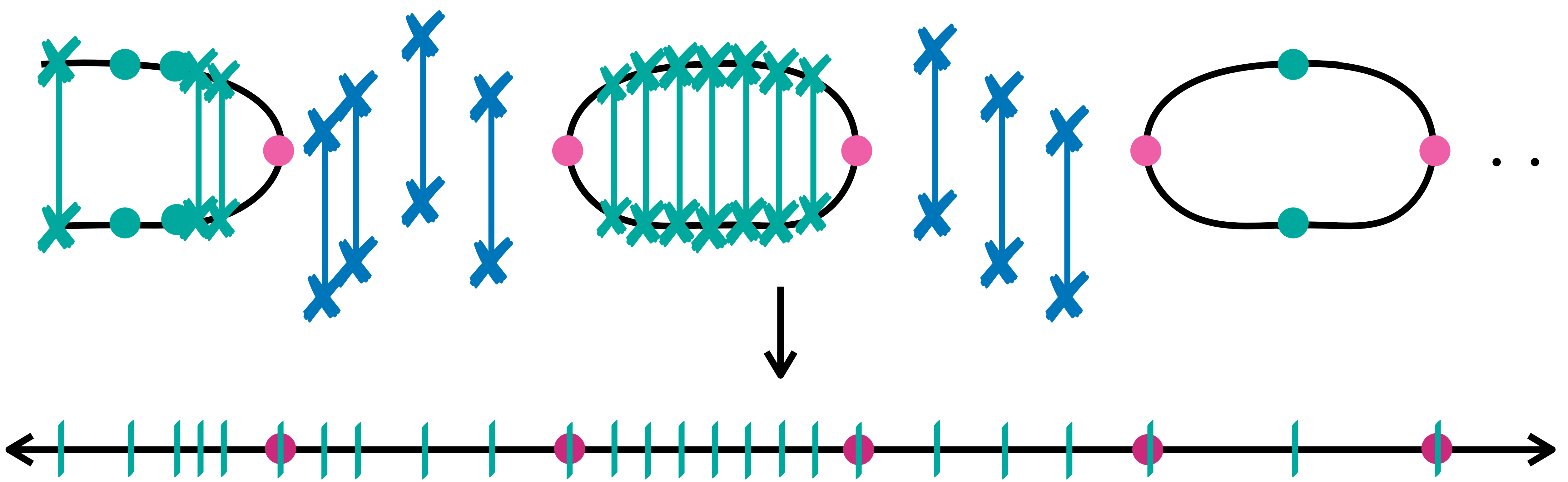


Can run a similar argument with a map to a genus 1 curve with infinitely many $\mathbb{Q}$-points!

Definition*: Let C be a nice algebraic curve over $\mathbb{Q}$.

A degree d point $x \in C(\overline{\mathbb{Q}})$ is AV–parametrized if x is parametrized by a positive rank abelian variety.

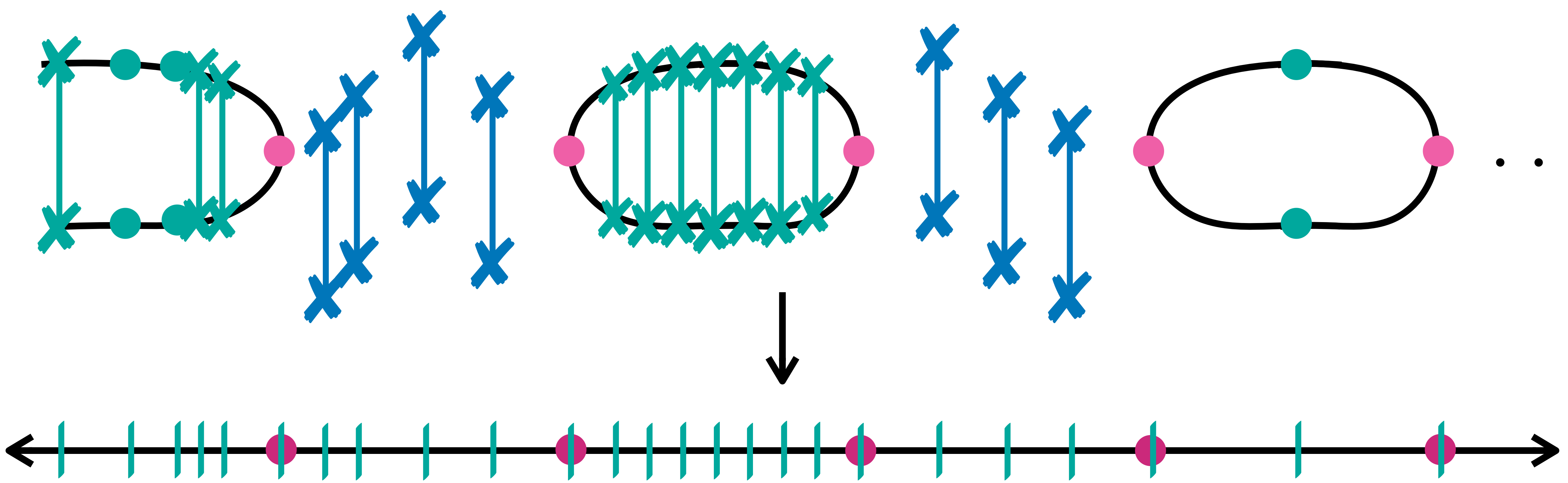
Otherwise, x is AV–isolated.



Precise Definition: Let C be a nice algebraic curve/$\mathbb{Q}$.

A degree d point $x \in C$ is AV–parametrized if there is a positive rank abelian variety $A \subset \operatorname{Pic}^0 C$ such that $[x] + A \subset W^d(C)$.

Otherwise, x is AV–isolated.



A point $x \in C$ is parametrized if it is $\mathbb{P}^1$- OR AV-parametrized.

It is isolated if it is NOT parametrized, i.e., it is $\mathbb{P}^1$- AND AV-isolated.

Parametrized vs. Isolated

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Parametrized points are "better understood". They cast shadows that can be detected by geometric techniques.

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Faltings's Theorem ('91) $+\epsilon$ (BELOV)

Let C be a nice algebraic curve over $\mathbb{Q}$. The curve C has infinitely many degree d points if and only if there is a degree d point that is $\mathbb{P}^1$– or AV–parametrized.

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Can make a geometric "only if" statement using "geometric shadows"

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Let C be a nice algebraic curve over $\mathbb{Q}$. The curve C has finitely many degree d points if and only if every degree d point is isolated.

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Probably the strongest statement that holds without assumptions on the curve

Let's focus on moduli spaces!

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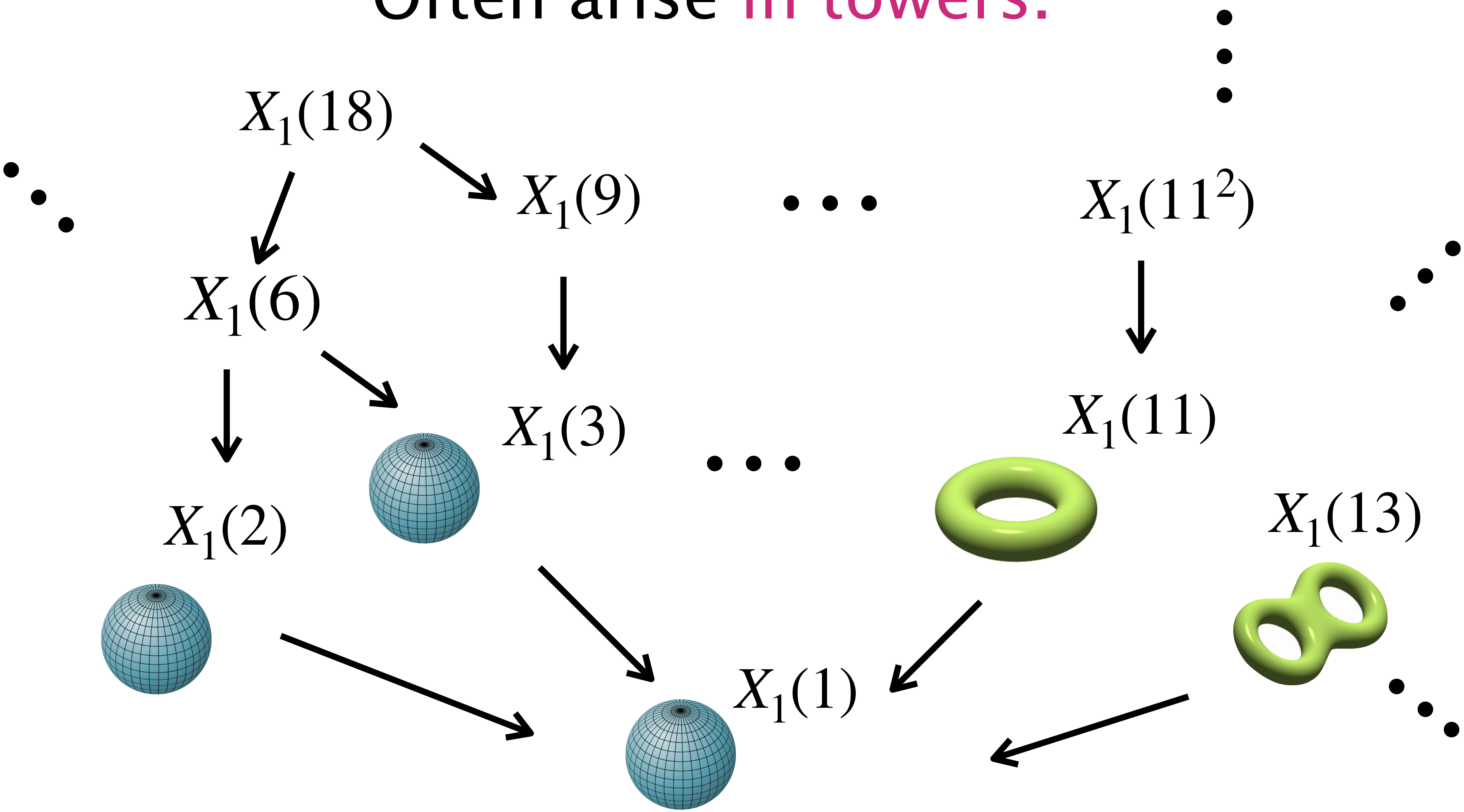
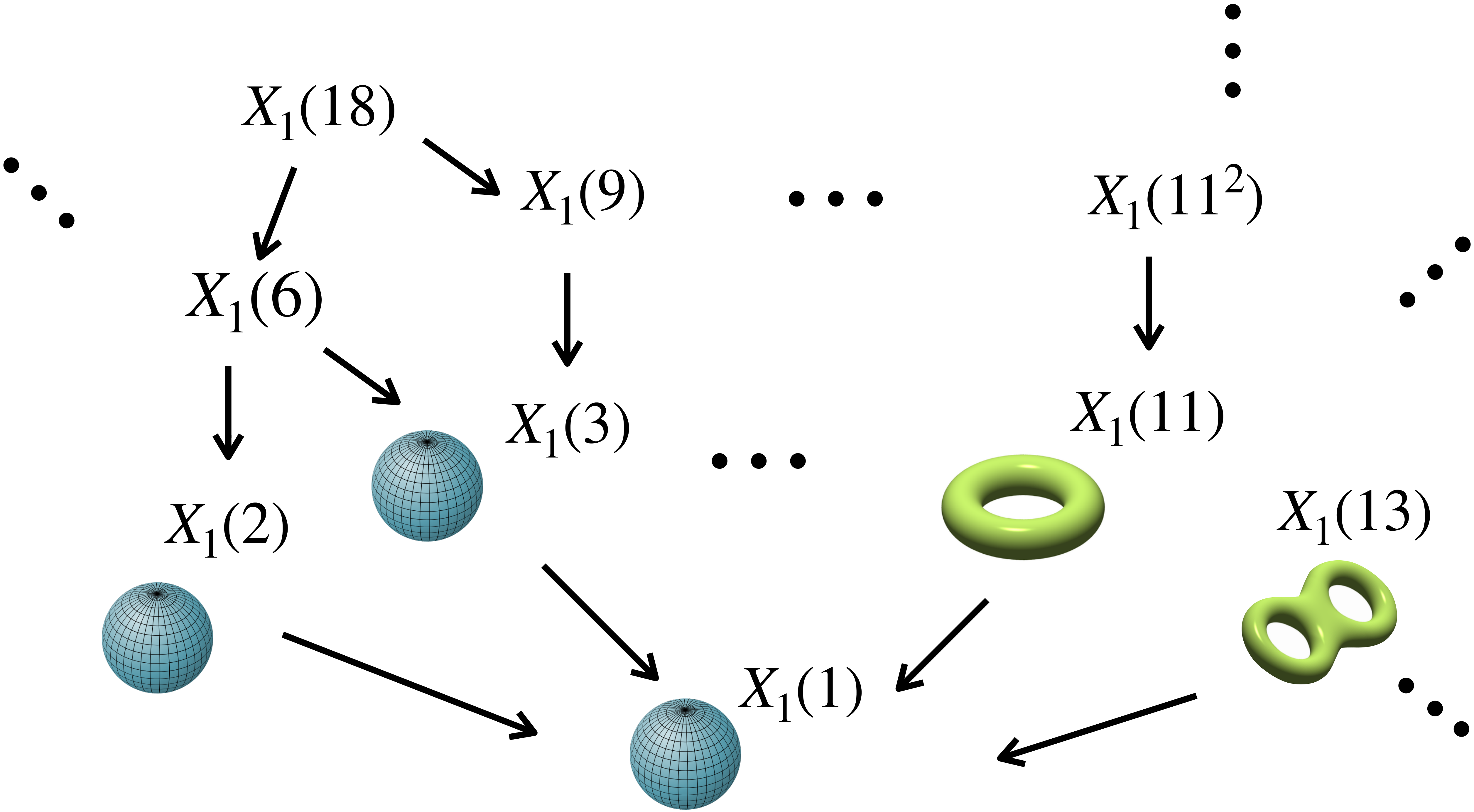
Often arise in towers.



How do isolated points behave in towers?



Let $f\colon C \to D$ be a morphism of curves.

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Then

$$x \begin{cases} \mathbb{P}^1 - \text{isolated} \\ AV - \text{isolated} \end{cases} \Rightarrow f(x) \begin{cases} \mathbb{P}^1 - \text{isolated} \\ AV - \text{isolated} \end{cases}.$$

... Then

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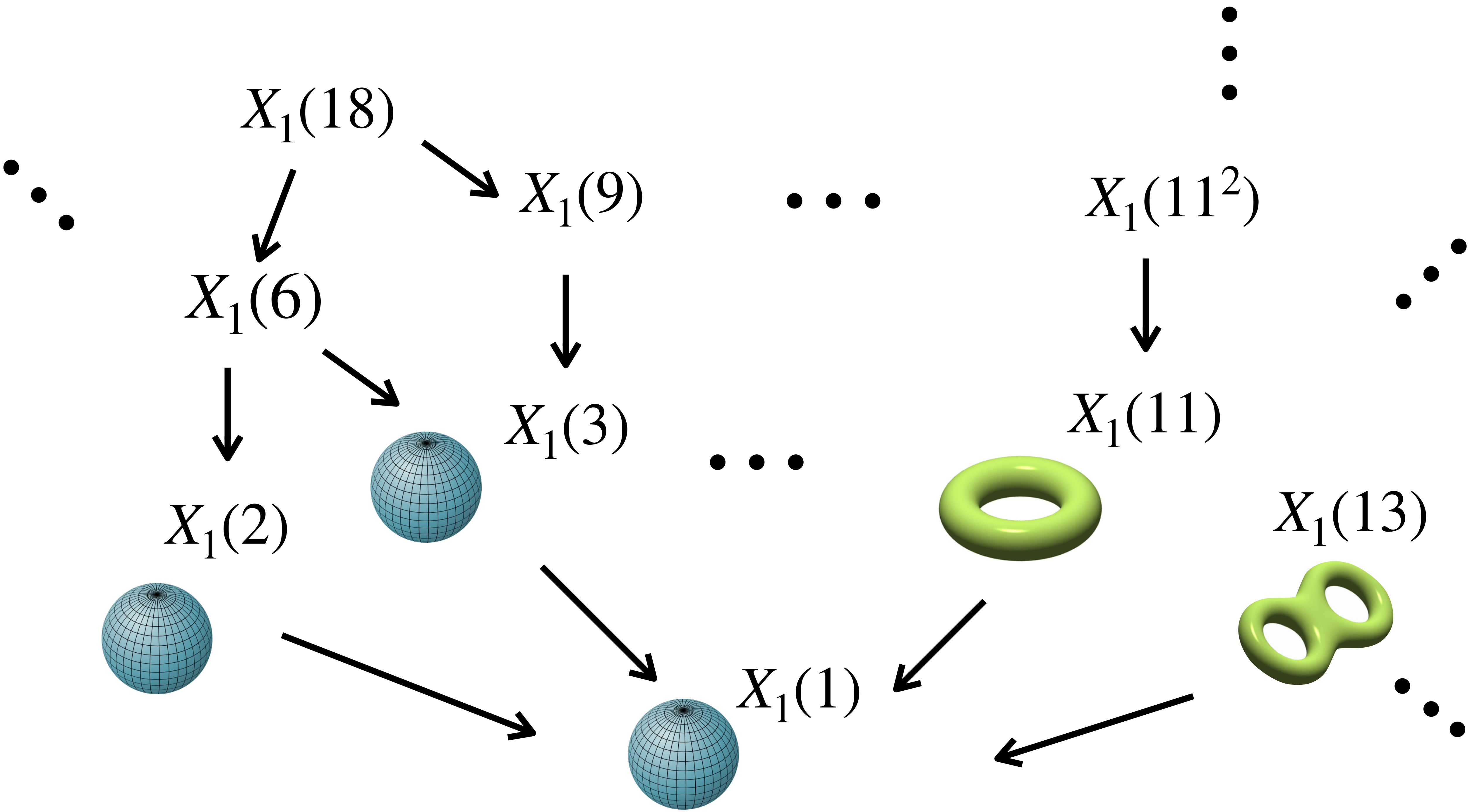

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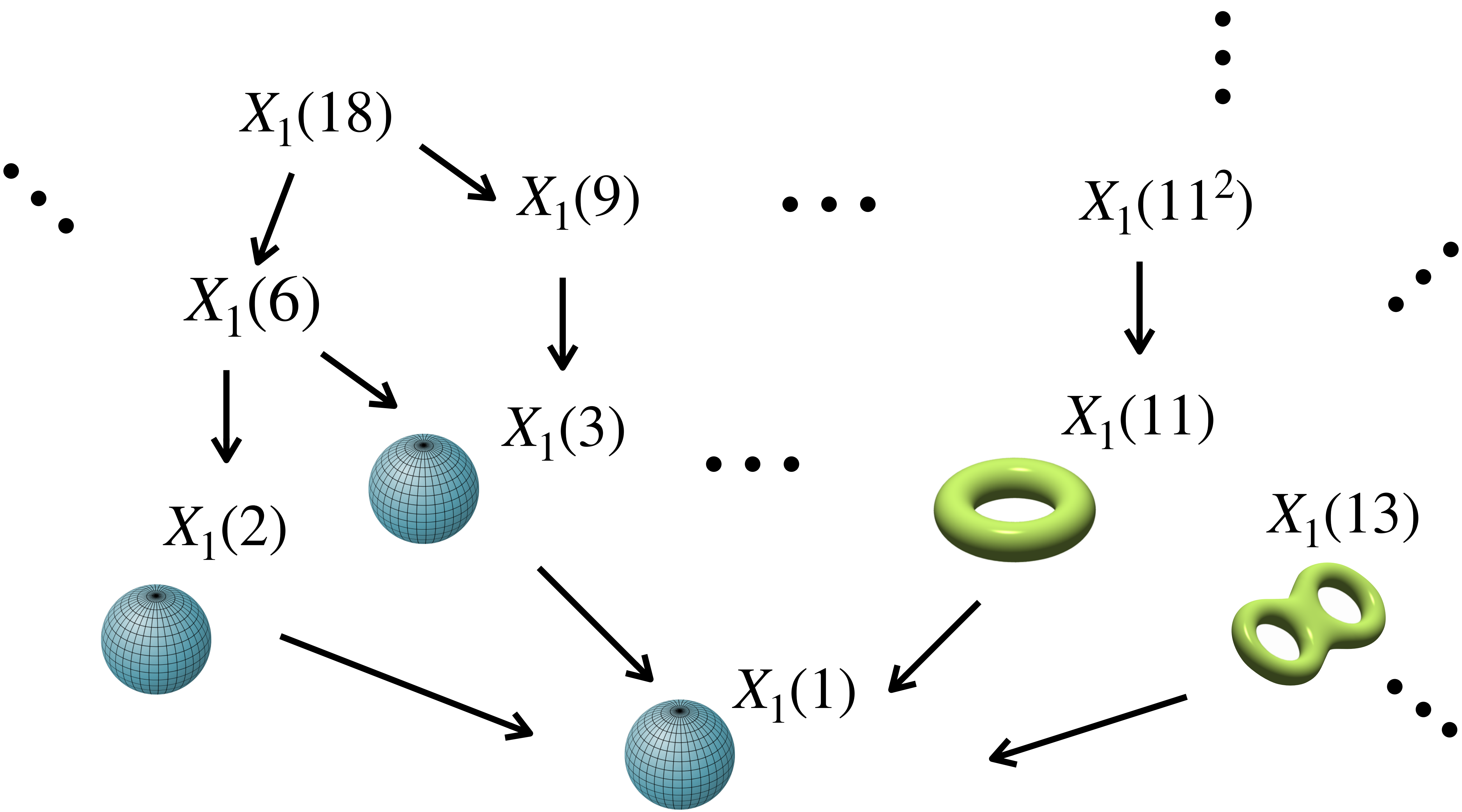
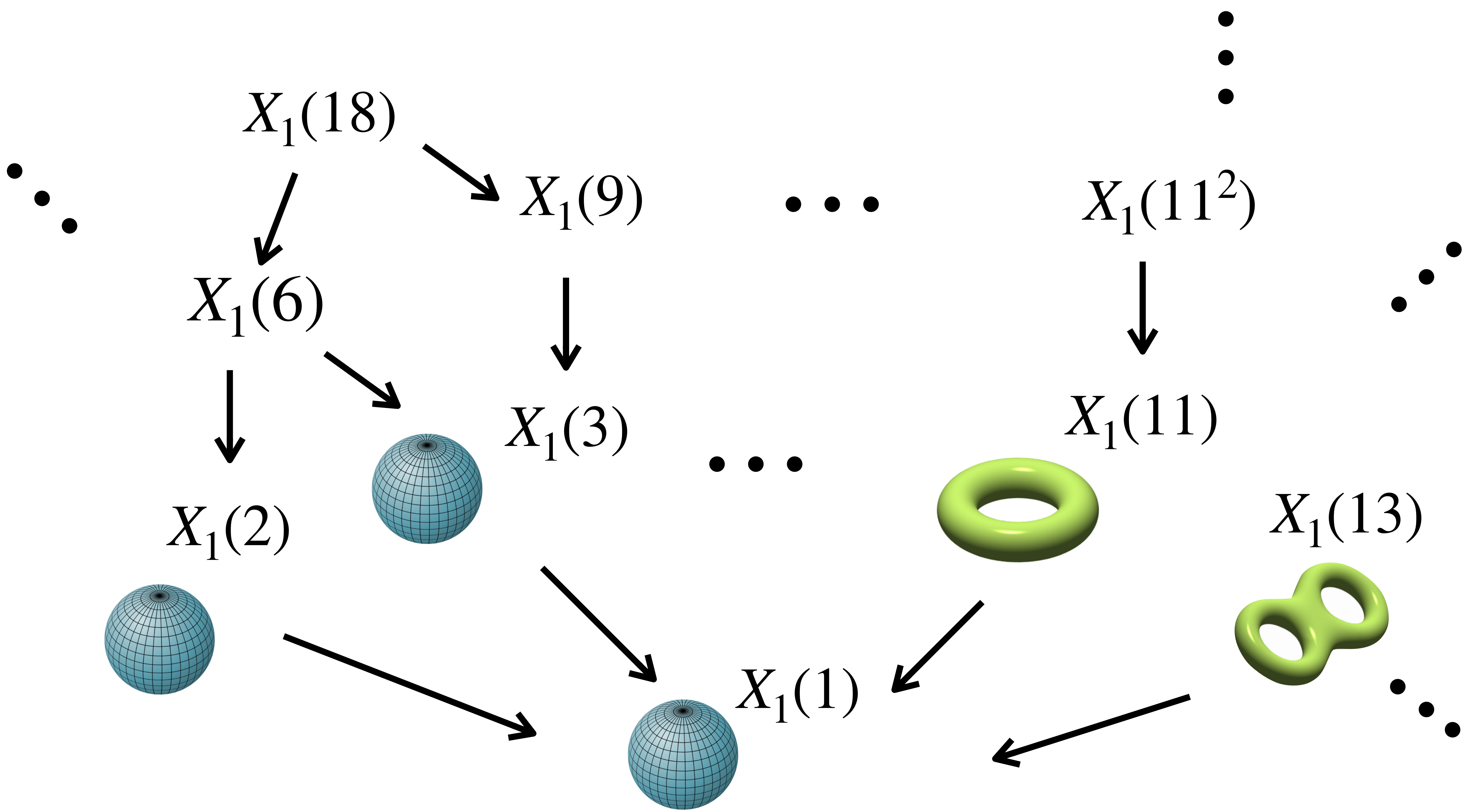
Consequence: The presence of isolated points can likely be detected low in the tower!



$X_1(18)$
$X_1(9)$
$X_1(11^2)$
$X_1(6)$
$X_1(3)$
$X_1(11)$
$X_1(2)$
$X_1(13)$
$X_1(1)$

(noncuspidal) points on $X_1(n)$ $\leftrightsquigarrow$ iso. classes of pairs (E, P), where P is a point of order n on E



Merel's Uniform Boundedness Theorem

(after Mazur, Kamienny)

Fix a positive integer d. There exists a constant $B(d)$ s.t. for all number fields $k/\mathbb{Q}$ of degree d and all elliptic curves E/k,

$$\#E(k)_{\mathrm{tors}} \leq B(d)$$

Consequence of
Merel's Uniform Boundedness Theorem
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Fix a positive integer d. There exists a constant $C(d)$ such that for all $n \geq C(d)$

$$x \in X_1(n) \text{ isolated} \Rightarrow \deg(x) > d$$

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Upshot: isolated points in the tower are computationally harder to detect

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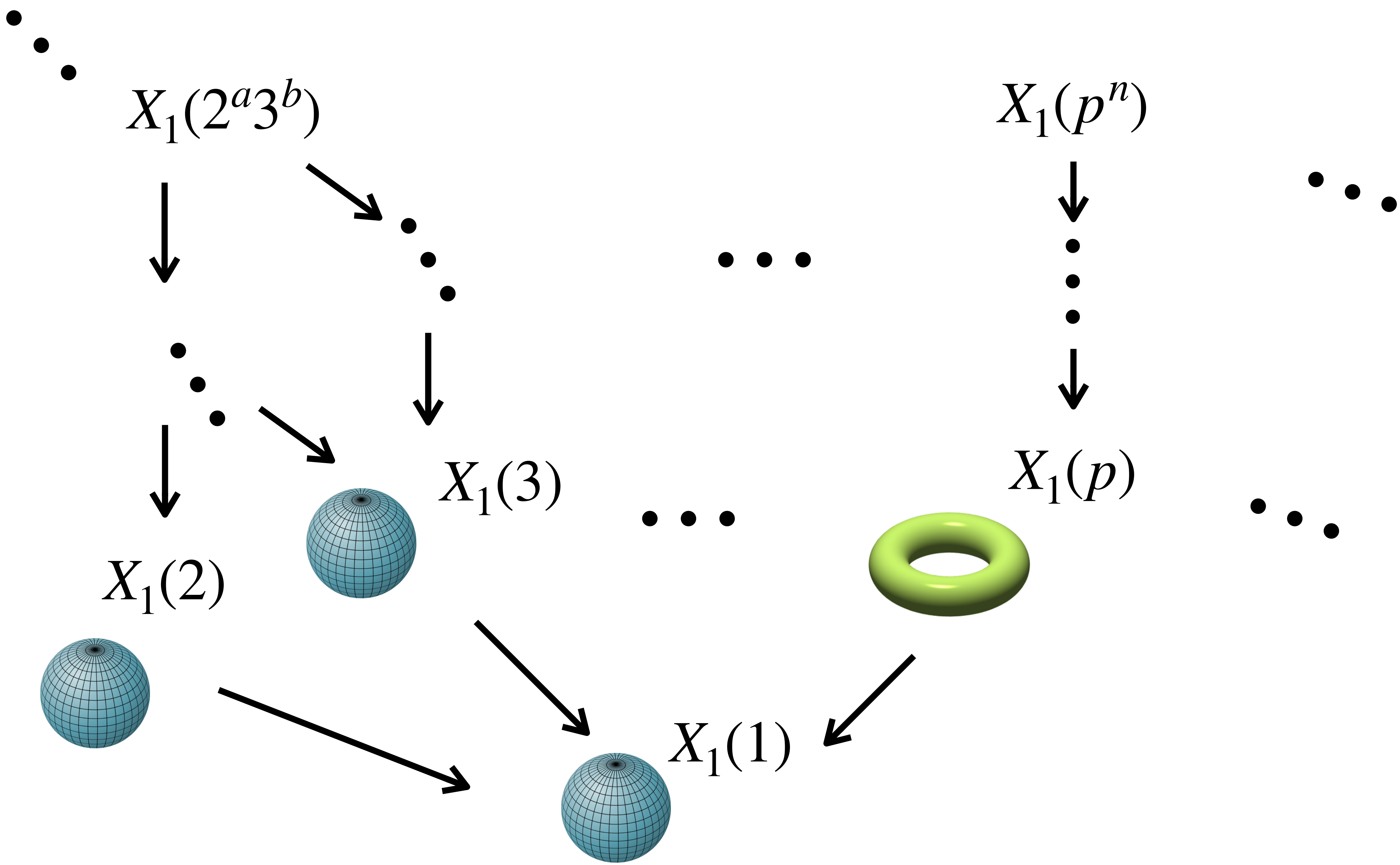
Consequences for $\{X_1(n)\}$



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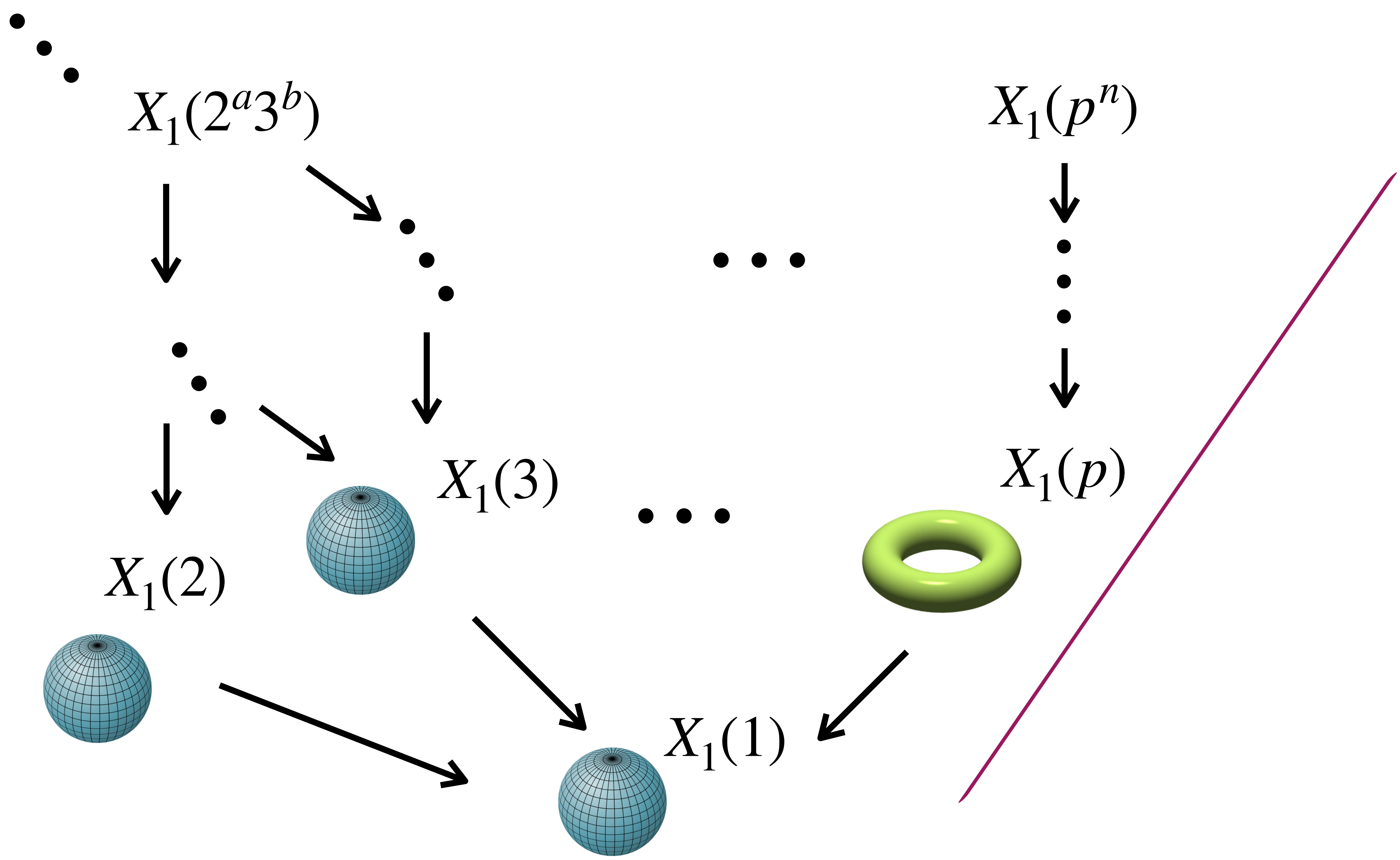
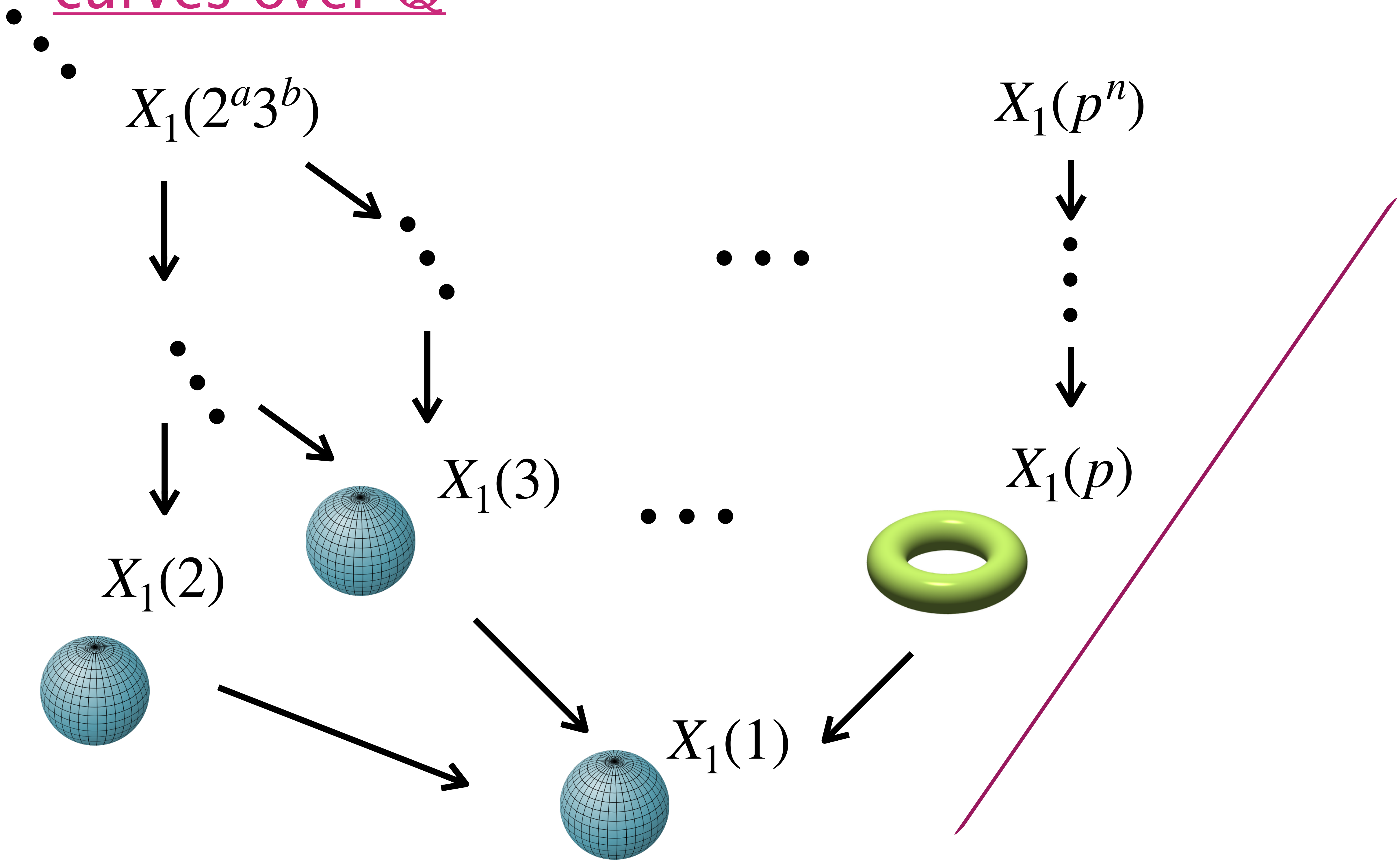
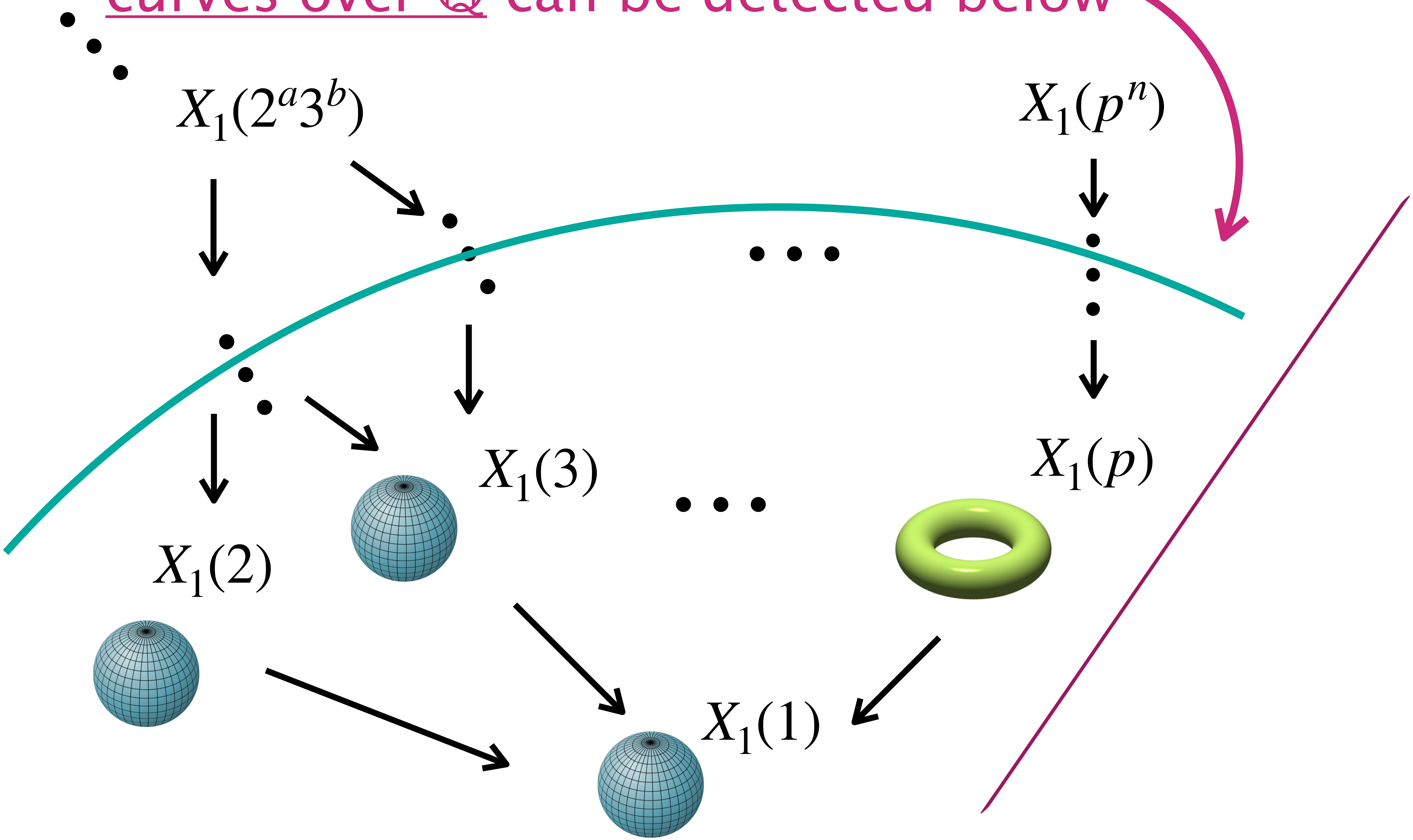


Non-CM isolated points on $\{X_1(2^a3^b\cdots p^n)\}$ that correspond to elliptic curves over $\mathbb{Q}$



Theorem (BELOV): Non-CM isolated points on $\left\{X_1(2^a3^b\cdots p^n)\right\}$ that correspond to elliptic curves over $\mathbb{Q}$ can be detected below



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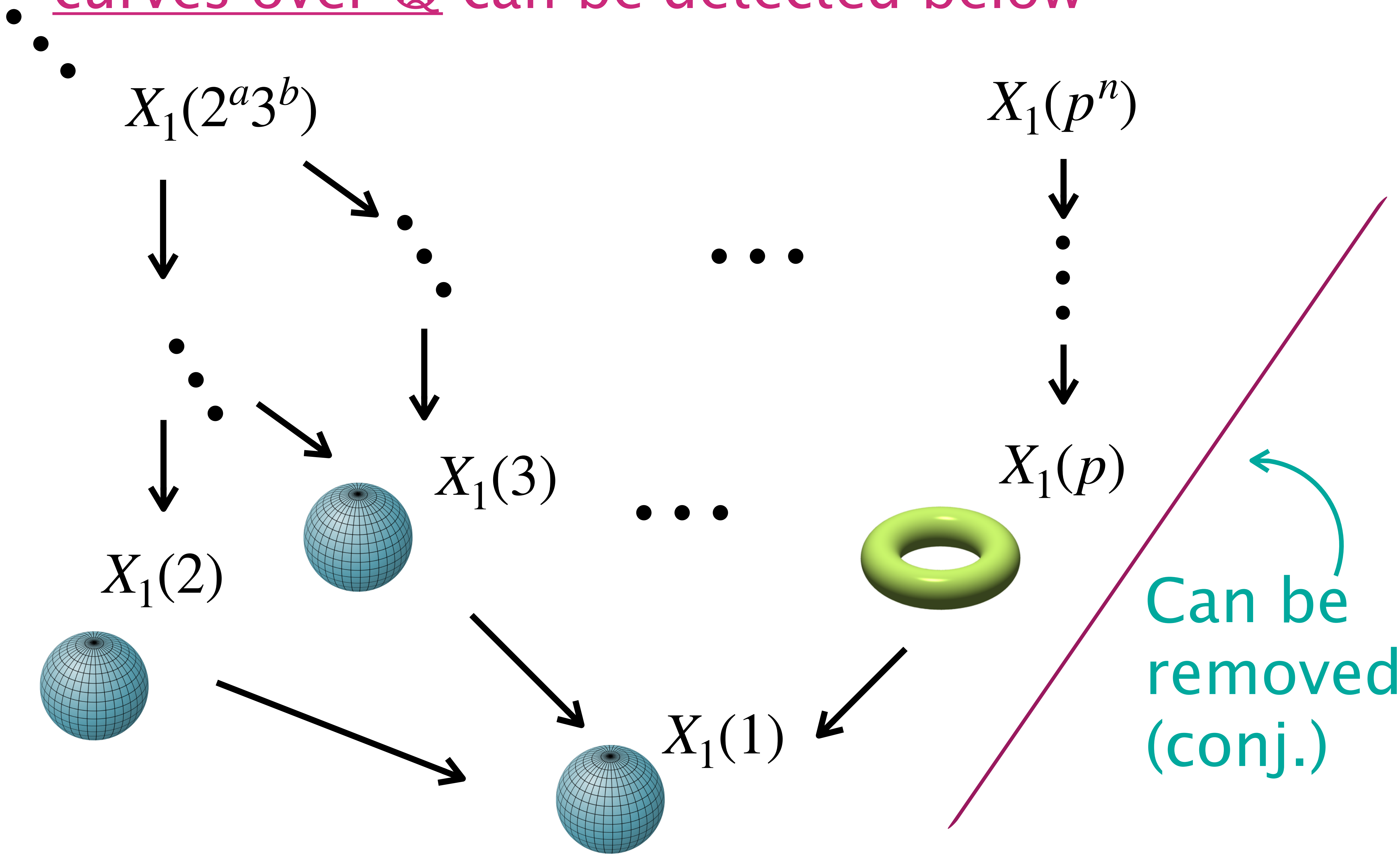

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