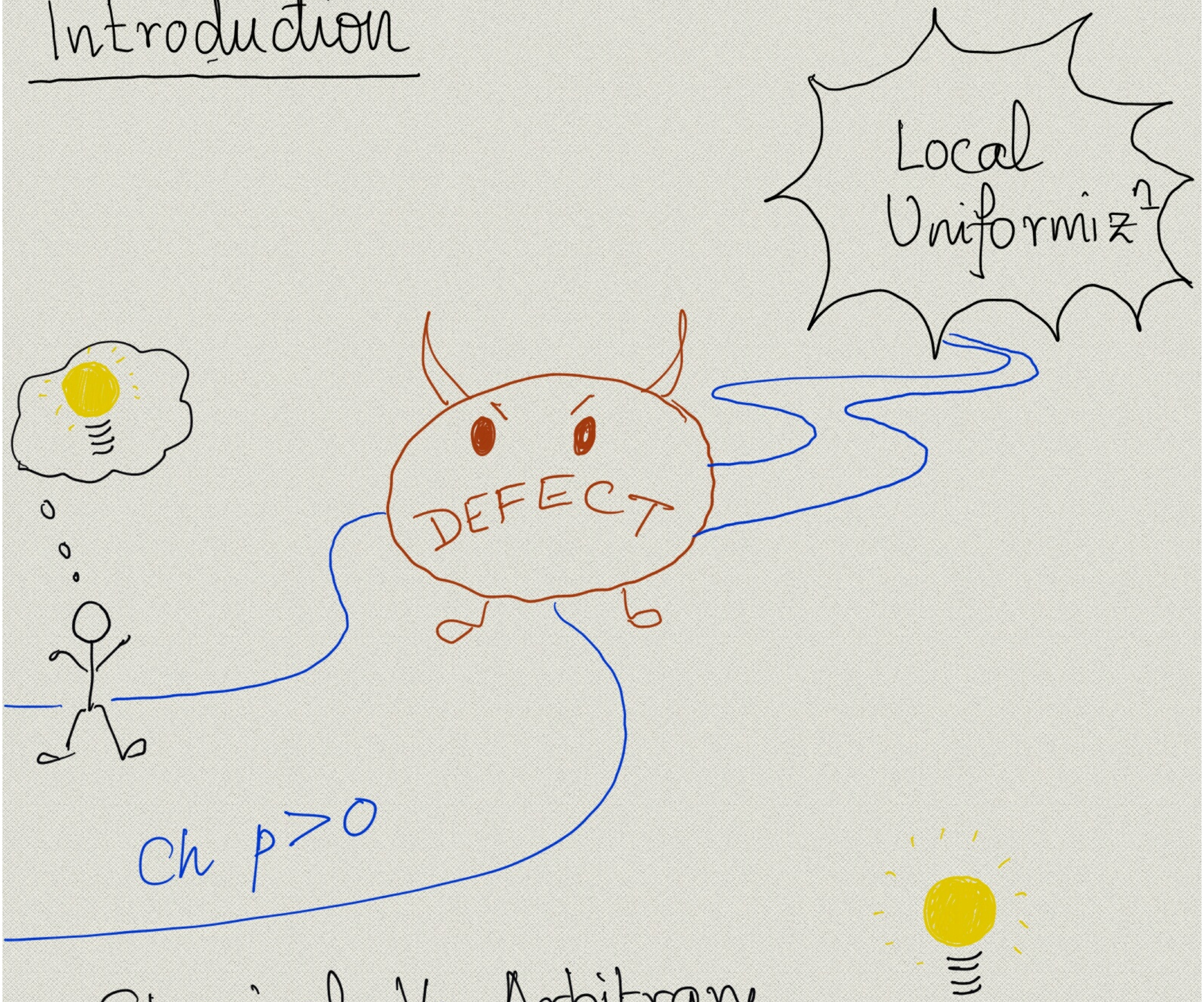
Upper Ramification Groups for Arbitrary Valuation Rings

Vaidehee Thatte
Joint with Kazuya Kato

Binghamton University
https://sites.google.com/view/vaideheethatte
CTNT Conference June 12-14, 2020

Introduction



- Classical Vs Arbitrary
- Ramification groups, $j(\sigma)$, Swan- Classical
- Degree p extensions in res. char $p>0$
- Upper ramification groups - Arbitrary

HOPE!

Comparison :

max id.	val. ring	val. field	val^n	res. field
$\mathfrak{m}_L \subset$	$B \subset$	L	ω	$l = B/\mathfrak{m}_L$
		Gal deg $n>1$	unique unique	
$\mathfrak{m}_K \subset$	$A \subset$	K	v	$k = A/\mathfrak{m}_K$
$v>0$	$v \geqslant 0$	Gal gp G	Ram^n index $e_{L\|K}$	Inertia deg $f_{L\|K}$

Classical: CDVR, k perfect

$(x) \subset k[[X]] \subset k((X))$ with X-adic

① $[L:K] = e_{L|K} \cdot f_{L|K}$

② Value group $\mathbb{Z}$ - Rank 1, ideals principal

③ $B = A[y]$; for some $y \in B$.

Arbitrary: K Henselian.

k can be imperfect, val gr can be arbitrary.

① $[L:K] = e_{L|K} \cdot f_{L|K} \cdot \boxed{d_{L|K}}$ $\to$ 1 ch0, $\to p^{(\cdot)}$ ch p

② Higher rank, non-discrete val. possible.

③ Fails.

Examples (deg p, Artin-Schreier)

① $K = \mathbb{F}_p((X))$, $L = K(\alpha)$ where $\alpha^p - \alpha = \frac{1}{X}$. Defectless ($e_{L|K} = p$)

$\beta^p - \beta = \frac{1}{X^p}$ gives the same ext^n.

$\frac{1}{X^p} = \frac{1}{X} + \left(\frac{1}{X}\right)^p - \frac{1}{X}$. & $\beta = \alpha + \frac{1}{X}$.

"Best f" is obtained in this way.

② $K = \bigcup_{r \in \mathbb{Z}_{\geq 0}} \mathbb{F}_p((X))(X^{1/p^r})$, $L = K(\alpha)$ where $\alpha^p - \alpha = \frac{1}{X}$. Has defect. Valgp $\mathbb{Z}\left[\frac{1}{p}\right]$

If we try the above process, we can get

$\frac{1}{X^{1/p}} \rightsquigarrow \frac{1}{X^{1/p^2}} \rightsquigarrow \dots$

Best f does not exist.

Classical Ramification Theory (CDVR, k perf)

Lower filtration: For $i \in \mathbb{Z}_{\geq -1}$,

$G_i := \{ \sigma \in G \mid \sigma b - b \in m_L^{i+1} \ \forall b \in B \}$

$G_{-1} = G \trianglerighteq \underset{\text{inertia sbgp}}{G_0} \trianglerighteq \cdots$ decreasing.

$G_i = \{1\}$ for i sufficiently large.

Breaks : $i \geq 0$ s.t. $G_i \neq G_{i+1}$.

Precisely $\{ \min_{b \in L^\times} \omega\left(\frac{\sigma b - b}{b}\right) \mid \sigma \in G_0, \sigma \neq 1 \}$

Upper filtration via Hasse-Herbrand f^n

$$\varphi_{L|K} : [-1, \infty) \longrightarrow [-1, \infty)$$
$$u \longmapsto \int_0^u \frac{dt}{(G_0 : G_t)}$$

$G_t = G_{[t]+1}$ for $t \notin \mathbb{Z}$.

$G^{\varphi(u)} := G_u$

Lower : adapted to subgroups.
Upper : adapted to quotients.

Hasse-Arf Thm For G abelian, jumps in $\{G^v\}$ are integers.

Log Lefschetz number $j(\sigma)$; $\sigma \in G\setminus\{1\}$

$$j(\sigma) := \min_{b \in L^\times} \left\{ \omega\left(\frac{\sigma b - b}{b}\right) \right\} \quad \in \mathbb{Z}_{\geq 0}$$

For degree p extensions, $j(\sigma)$ is independent of choice of σ

Swan Conductor For a finite dimensional rep ρ of G over ch 0 field.

$$Sw(\rho) := \frac{1}{e_{L|K}} \sum_{\sigma \in G\setminus\{1\}} j(\sigma)\left[\dim\rho - \mathrm{Tr}\,\rho(\sigma)\right]$$

For degree p extensions & $\rho : G \to \mathbb{C}^\times$

$$Sw = \frac{p}{e_{L|K}} j(\sigma)$$

$j(\sigma)$ & Sw def$^{\underline{s}}$ work when k is imperfect.

We generalize these to arbitrary valuation rings (deg p case).

Degree p Extensions, $\operatorname{ch} k = p > 0$

* Results in equal characteristic are below. Analogs are true in mixed characteristic $(0,p)$.

K Henselian, $L|K$ Artin-Schreier extⁿ

"Replace numbers by ideals"

$J_\sigma := \left\langle \frac{\sigma b - b}{b} \;\middle|\; b \in L^\times \right\rangle$ of B.

$\mathcal{H} := \left\langle \frac{1}{f} \;\middle|\; L = K(\alpha);\ \alpha^p - \alpha = f \right\rangle$ of A.

<u>Propⁿ (T)</u> J_σ is principal

$\iff L|K$ is defectless*

$\iff \mathcal{H}$ is principal

In this case*, $J_\sigma = (1/\alpha)$, $\mathcal{H} = (\frac{1}{f})$ where f is "best"

<u>Thm (T)</u> $\mathcal{H} = \langle N_{L|K}(J_\sigma) \rangle$

In example 1: $\mathcal{H} = (X)$, $J_\sigma = (1/\alpha)$

$N(1/\alpha) = 1/N(\alpha) = X$.

In example 2: $\mathcal{H}$ & J_σ are infinitely generated.

Upper Ramification Groups - Arbitrary

(Preprint with K. Kato 2019)

A: Henselian val^n ring (not a field)

K: $Q(A)$. $\overline{K}$ = sep. closure

$\overline{A}$ = int. cl. of A in $\overline{K}$ (also val^n ring)

$G = \mathrm{Gal}(\overline{K}|K)$.

Def^n For nonzero proper ideal I of $\overline{A}$, define a closed normal subgroup $G^I_{\log}$ (resp.ly G^I_{nlog}) to be the intersection of the kernels of $G \longrightarrow \mathrm{Gal}(L|K)$ where L ranges over all finite Galois extensions of K in $\overline{K}$ with "ramification logarithmically (resp. non-logarithmically) bounded by I."

- $G^I_{log} \subset G^I_{nlog}$
- $G^I_* \supset G^J_*$ if $I \subset J$
- Compatible for DVRs

k can be imperf

Abbes-Saito: G^r_{log}, G^r_{nlog} ; $r \in \mathbb{Q}_{>0}$

Classical : G^r_{cl}

(i) $G^r_{cl} = G^r_{log} = G^{r+1}_{nlog}$

(ii) $G^r_{nlog} =$ inertia sbgp for $0 < r \leq 1$.

(iii) $G^r_* = G^I_*$ for $I = I(r)$;

$$I(r) = \{ x \in \bar{A} \mid \mathrm{ord}_{\bar{A}}(x) \geq r \}$$

and

$$G^I_* = \overline{\bigcup_{\substack{r \in \mathbb{Q}_{>0} \text{ s.t.} \\ I(r) \subset I}} G^r_*}^{\,\text{cl. in } G}$$

Thm (Kato, T.) Assume that $\operatorname{ch} k = p > 0$ and consider a degree p extension L. Then for a nonzero proper ideal I of $\bar{A}$, the image of $G_{\log}^{I}$ in $\mathrm{Gal}(L|K)$ is $\mathrm{Gal}(L|K) \iff I \cap A \supset \mathcal{H}$

and is $\{1\} \iff I \cap A \subsetneq \mathcal{H}$

Rem: $\mathcal{H}$ generalizes Swan.

Break occurs at Swan in the classical case.

Thm (Kato, T.) Let $L|K$ be a defectless finite Galois extension. Then any break I of $L|K$ (I s.t. $\mathrm{Gal}(L|K)_{\log}^{I} \neq \mathrm{Gal}(L|K)_{\log}^{J}$ $\forall$ nonzero $J \subsetneq I$) is a principal ideal.

Current project: Converse.

Thank You!