CONSTRUCTING GENUS g CURVES OF RANK $\geqslant 4g+15$

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THEOREM :

Let $g \geqslant 8$, $g \equiv 2 \pmod 3$.
Then, there exist infinitely many genus g curves $X/\mathbb{Q}$, pair-wise non-isomorphic $/\overline{\mathbb{Q}}$, which satisfy

$$\operatorname{rank} X/\mathbb{Q} \geqslant 4g+15$$

THE MAIN ACTORS:

- K : a field
- Curve / K : smooth, projective, geometrically integral, 1 dim'l scheme / K (variety / K)
- Jacobian of a genus g curve X/K:
  - Abelian variety $J(X)/K$ of dim. g
  - Assume $X(K) \neq \phi$. Then

$$J(X)(L) = \mathrm{Pic}^0(X_L)$$

Group of degree 0 divisors on X_L, modulo principal divisors.

Base-change of X to L.

EXAMPLES: (Assume $X(K) \neq \phi$)

$g = 0$: $X \cong \mathbb{P}^1_K$, $J(X)$ is a point.

$g = 1$: $X \cong J(X)$ (elliptic curve/K)

MORDELL - WEIL THEOREM :

If K is a number field, then

$$J(X)(K) \cong \mathbb{Z}^r \times A$$

finite abelian group

DEFINITION : rank $X/K := r$.

QUESTION : Fix $g \geq 1$. Does rank $X/\mathbb{Q}$ remain bounded as $X/\mathbb{Q}$ varies?

Equivalently, does the following constant $R(g)$ exist?

$R(g)$:= smallest integer such that

$$\text{rank } X/\mathbb{Q} \leq R(g)$$

for all but finitely many $X/\mathbb{Q}$ of genus g.

⚠ $X/\mathbb{Q}$, $Y/\mathbb{Q}$ are regarded as the "same curve" if $X_{\overline{\mathbb{Q}}} \cong Y_{\overline{\mathbb{Q}}}$.

$$\boxed{\begin{array}{c} R(g) \geq c \\ \Updownarrow \\ \exists \text{ infinitely many genus } g\ X/\mathbb{Q}, \\ \text{pair-wise non-isomorphic} / \overline{\mathbb{Q}}, \text{ s.t.} \\ \operatorname{rank} X/\mathbb{Q} \geq c. \end{array}}$$

For all $g \geq 1$, existence of $R(g)$ is open.

- $g = 1$: Multiple folklore conjectures!

  Heuristics suggest: $R(1) \leq 21$.

  - Watkins, 2015
  - Park - Poonen - Voight - Wood, 2019

- $g \geq 2$: No folklore conjectures / heuristics.

RANK RECORDS :

- $R(1) \geq 19$ (Elkies, 2006)
- $R(2) \geq 28 + 4$ (Kulesz, 2001) +
- $R(3) \geq 26$ "
- For $g \geq 4$:

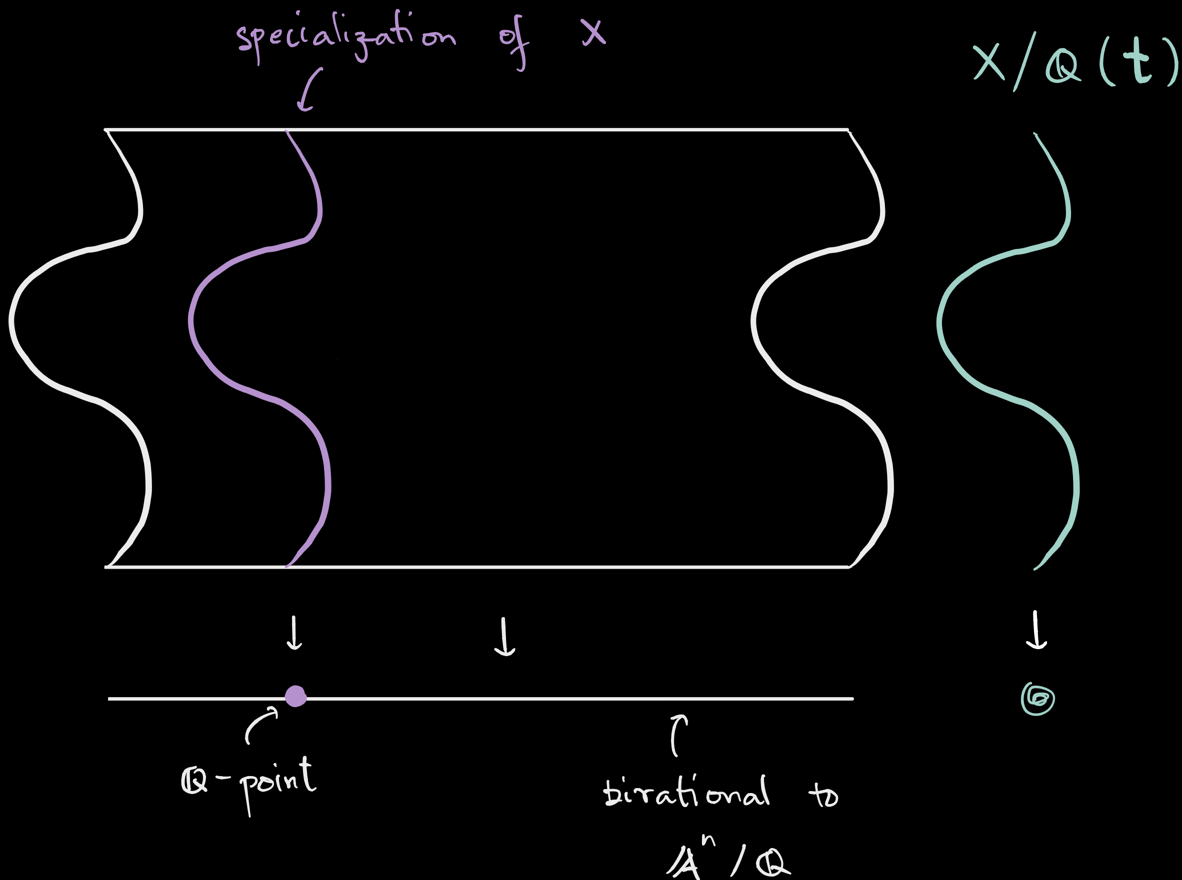
  $R(g) \geq 4g + 7$ (Shioda, 1998)

TODAY : New records !

$$R(g) \geq \begin{cases} 30 & \text{if } g = 5 \\ 4g + 8 & \text{if } g \equiv 1 \ (3), \ g \geq 4 \\ 4g + 15 & \text{if } g \equiv 2 \ (3), \ g \geq 8 \end{cases}$$

SPECIALIZATION METHOD:

- Construct a genus g $X/\mathbb{Q}(\mathbf{t})$ ($t_1, \ldots, t_n$)
  with many $\mathbb{Q}(\mathbf{t})$-points. (MESTRE)
- Show that these generate
  a rank C subgroup of $J(X)(\mathbb{Q}(\mathbf{t}))$



NÉRON'S THEOREM $\Rightarrow$ $R(g) \geqslant C$

REMARK :

(Hammonds, Kim, Logsdon, Lozano-Robledo, Miller, 2020) prove :

$$R(g) \geq 4g + 2,$$

conditional on a conjecture of Nagao.
(Without producing explicit points!)

KEY LEMMA : K a field, $\operatorname{char} K \neq 2$.
Given monic, deg. $2k$ $m(x) \in K[x]$,
$\exists!$ $h(x), \ell(x) \in K[x]$ s.that :

"sq-root" approximation

"sq-root" remainder

$$m(x) = h(x)^2 - \ell(x)$$

monic, deg. $2k$

monic, deg k

deg. $\leq k-1$.

"generically": deg. $\ell(x) = k-1$

(open condition on coeff's of $m(x)$

EXAMPLE :

$$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 - \left(\frac{a^2}{4} - b\right)$$

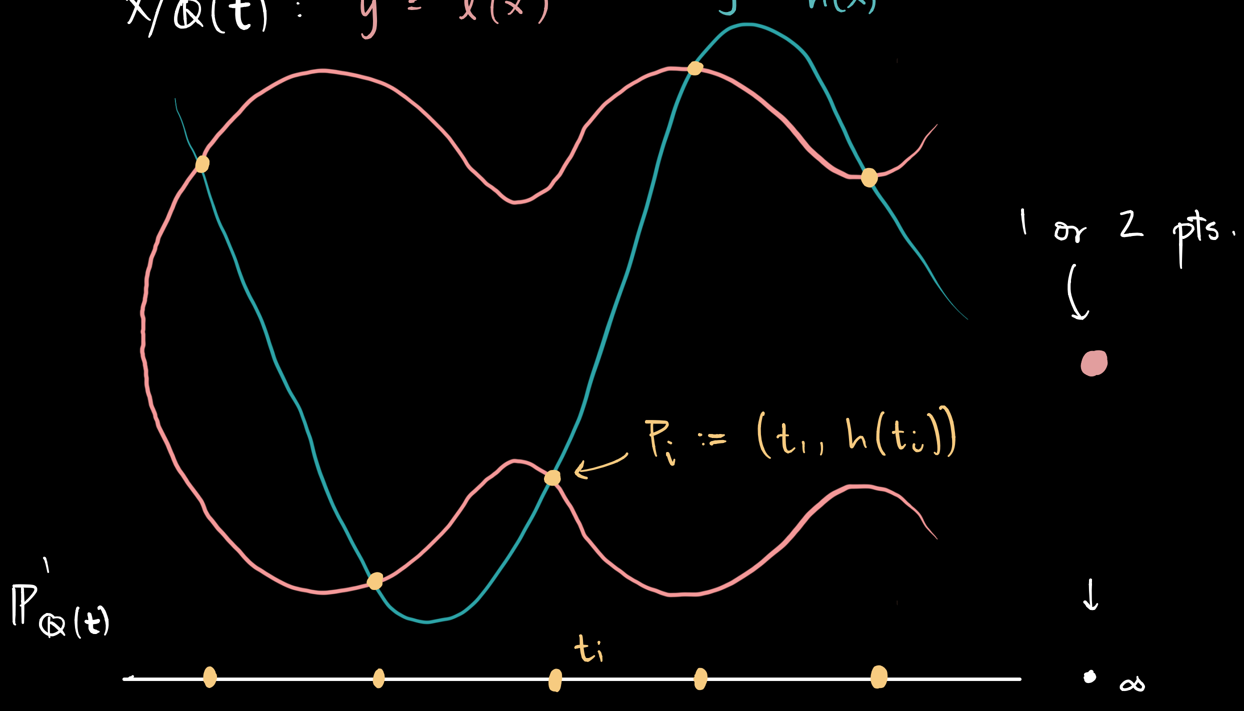
" Completing the square "

MESTRE'S CONSTRUCTION :

$$t_1, \ldots, t_{2k} \rightsquigarrow (x - t_1) \cdots (x - t_{2k})$$

$$h(x)^2 - \ell(x) \overset{\text{Lem.}}{=} m(x)$$

$X/\mathbb{Q}(t)$: $y^2 = \ell(x)$ $\qquad$ $y = h(x)$



$$P_1, \ldots, P_{2k} \in X(\mathbb{Q}(t))$$

$$\rotatebox{90}{$\rightsquigarrow$}$$

$$[D_i] := [2P_i - D_\infty] \in J(X)(\mathbb{Q}(t))$$

pre-image of ∞, deg. 2 divisor on X

$$\operatorname{div}\left((y - h(x))^2\right) = \sum_{i=1}^{2k} 2P_i - D_\infty$$

$$\Downarrow$$

$$[D_1] + \cdots + [D_{2k}] = 0 \in J(X)(\mathbb{Q}(t))$$

<u>THEOREM (Shioda, 1995)</u>:

$$\operatorname{rank} X/\mathbb{Q}(t) \geq 2k - 1$$

<u>FACT</u>: Genus of $X/\mathbb{Q}(t)$: $y^2 = \ell(x)$:

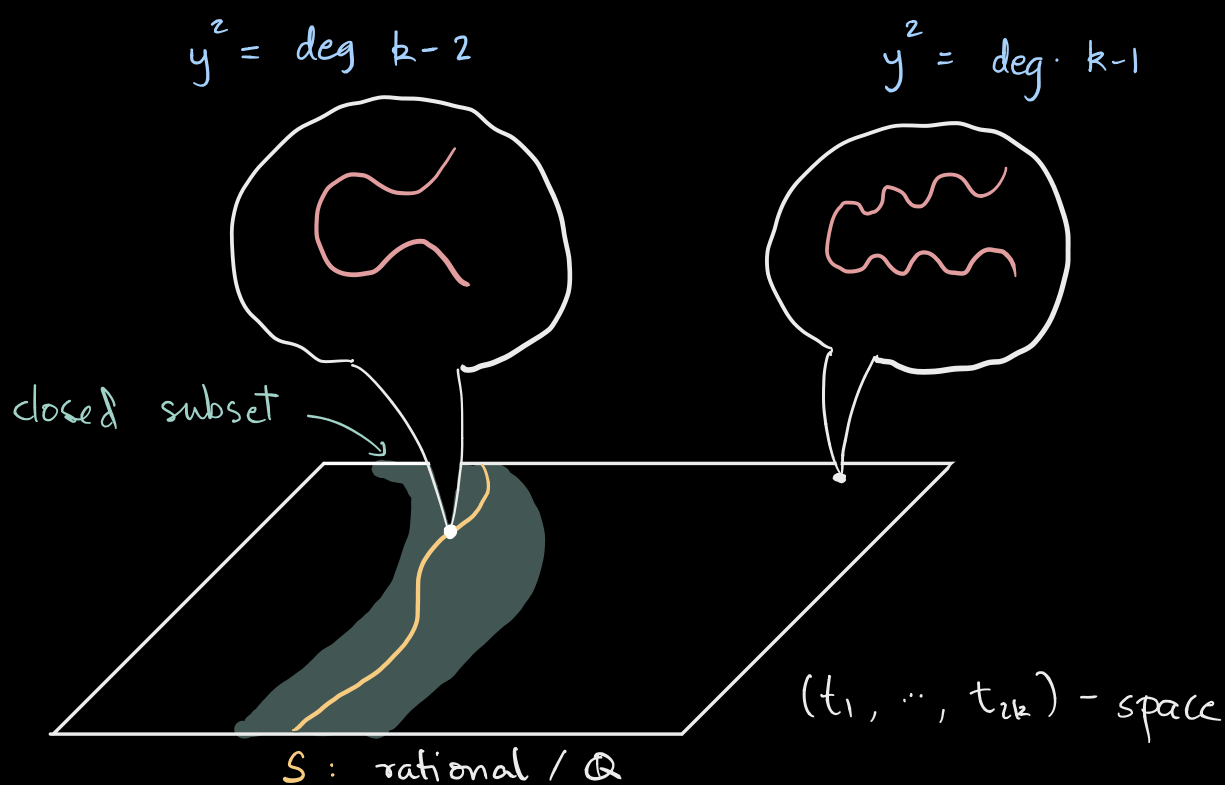
$$g = \frac{\underline{\deg}\ \ell(x) - 1}{2} \quad \text{or} \quad \frac{\underline{\deg}\ \ell(x) - 2}{2}$$

$$\frac{k-2}{2} \quad \text{or} \quad \frac{k-3}{2}$$

$$(\text{Néron}) \Rightarrow R(g) \geq 4g + 5$$

Q : How to improve on this ?

IDEA: Find a rational $\mathbb{Q}$-variety $S \subset \mathbb{A}^{2k}$ on which the specializations degenerate to lower genus curves.



HOPE : get same rank $2k-1$ for

$$\tilde{X}/\mathbb{Q}(S) : y^2 = \ell(x) \leftarrow \deg \leq k-2$$

But ... how to find such an S ?

OBSERVATION : Set $k = de$. Suppose

$a_1, \dots, a_{2de}$ is $(e, 2d)$ - composite :

$$\prod_{i=1}^{2de} (x - a_i) = \underbrace{m}_{\text{monic, deg. } 2d}(\underbrace{g(x)}_{\text{monic, deg. } e})$$

Then : $$m(g(x)) \overset{\text{Lem.}}{=} \underbrace{h}_{\text{monic, deg. } d}(g(x)) - \underbrace{\ell}_{\text{deg } d-1}(g(x))$$

$$\deg \ell(g(x)) = e(d-1) \quad (\text{generically})$$
$$= (ed - 1) - \underbrace{(e-1)}_{\text{drop in degree!}}$$

ACCOUNTING :

$\tilde{X}/\mathbb{Q}$: $y^2 = \ell(g(x))$, genus $\tilde{g}$:

$$\boxed{\text{Hope} : \operatorname{rank} \tilde{X}/\mathbb{Q} \geq 4\tilde{g} + 2e + 3}$$

(assume $2d \geq 8$)

EXAMPLE: Take $S = \mathbb{A}^{2d} \subseteq \mathbb{A}^{4d}$:

$(t_1, -t_1, \ldots, t_{2d}, -t_{2d})$ is $(2, 2d)$-comp:

$$\prod_{i=1}^{2d} (x \pm t_i) = m(x^2) \quad \text{monic deg. } 4d$$

$$= \underbrace{h(x^2)^2}_{\text{monic deg. } 2d} - \underbrace{\ell(x^2)}_{\text{deg. } 2d-1-1}$$

$$\rightsquigarrow$$

$t_1, \ldots, t_{2d}$

$\tilde{\Gamma} / \mathbb{Q}(\mathbf{t}) : y^2 = \ell(x^2)$, genus $\tilde{g}$ $\quad (\geq 2)$

$$\tilde{\Gamma} \cap \{y = h(x^2)\} = \{4\tilde{g} + 8 \ \mathbb{Q}(\mathbf{t})\text{-pts.}\}$$

THEOREM (Shioda, 1998):

$$\operatorname{rank} \tilde{\Gamma} / \mathbb{Q}(\mathbf{t}) \geq 4\tilde{g} + 7$$

So, by Néron's theorem, get:

$$R(g) \geq 4g + 7$$

To improve bound: find S that parametrizes $(6, 2d)$-comp. tuples:

Let $L = \mathbb{Q}(\zeta)$ (3^{rd} cyclotomic field).

$S \subset (s_1, t_1, \ldots, s_{2d}, t_{2d})$-space:

$$N_{L/\mathbb{Q}}(s_1 + \zeta t_1) = \cdots = N_{L/\mathbb{Q}}(s_{2d} + \zeta t_{2d})$$

- $S \overset{\text{open}}{\supset} \mathrm{Res}_{L/K}\, \mathbb{G}_{m,L} \times \left(\mathrm{Res}^1_{L/K}\, \mathbb{G}_{m,L}\right)^{2d-1}$

(Voskrosenskii) $\Rightarrow$ S is rational / $\mathbb{Q}$!

- $$\prod_{i=1}^{2d} (x \pm s_i)(x \pm t_i)(x \pm (s_i + t_i))$$

$\Big\|$ (Gloden, 1944)

monic, deg. 2d

$$m(g(x))$$

monic, deg. 6

$$\| $$

$$h(g(x))^2 - \ell(g(x))$$

$\Theta / \mathbb{Q}(s) \quad : \quad y^2 = \ell(g(x))$

$\deg \; 6(d-1), \quad \text{genus } \tilde{g} = 3d-4$

THEOREM:

$d = 2$, $\tilde{g} = 2$: rank $\Theta / \mathbb{Q}(s) \geq 16$

$d = 3$, $\tilde{g} = 5$: rank $\Theta / \mathbb{Q}(s) \geq 30$

$d \geq 4$, $\tilde{g} \geq 8$: rank $\Theta / \mathbb{Q}(s) \geq 4\tilde{g} + 15$

$$\Big\Downarrow \; (\text{Néron})$$

$$\boxed{\text{For } g \geq 8,\; g \equiv 2\,(3),\; R(g) \geq 4g + 15}$$

OPEN QUESTION :

or $n \geq 2$, $e \geq 13$

Let $n \geq 4$, $e \geq 7$. Does $\exists$ an (e,n)-composite tuple

$$(t_1, \ldots, t_{en}) \in \mathbb{A}^{en}(\mathbb{Q})?$$

(See : PROUHET - TARRY - ESCOTT PROBLEM)

Equivalently : Does there exist a degree ≥ 7 morphism $\mathbb{A}^1 \to \mathbb{A}^1$ that splits / ≥ 4 $\mathbb{Q}$-points ?

THANK YOU !
