Moduli of Galois Representations (CTINT 2020)

Notation:
- \( p \) a prime
- \( K/\mathbb{Q}_p \) a finite extension
- \( G_K := \text{Gal}(K/K) \), the absolute Galois gp. of \( K \).

Theorem: (Emerton-Gee, 2019)
For each \( d \geq 1 \), there exists a finite type algebraic stack \( \mathfrak{F}_d \) over \( \overline{\mathbb{F}_p} \) with the following properties:
(1) For each finite $F/F_p$

$\overline{F}_p$-pts of $\mathcal{X}_d \leftrightarrow$ continuous reps $\overline{\rho}:G_k \to GL_d(\overline{F})$

(2) $\overline{\mathcal{X}}_d$ is equidimensional of dim. $[K:Q_p]^d(2)$.

(3) A description of the irreducible components of $\overline{\mathcal{X}}_d$, namely:

Each component $\mathcal{X}(5) \subseteq \overline{\mathcal{X}}_d$ has a dense open subset $U \subseteq \mathcal{X}(5)$ whose $F$-pts. are certain specified...
Successive ext’s of characters.

Application: (Emerton-Gee)
Every $\overline{\rho}$ as in (1) lifts to characteristic 0.
(Previously known for $d \leq 3$.)

Today (joint with Caraiani, Emerton, Gee): a more precise description of the components of $\overline{\mathcal{F}_2}$ ($p > 2$).

To each $\overline{\rho}: \mathbb{G}_k \to \text{GL}_2(\mathbb{F}_p)$ there is an associated set
$W(p)$ of irreducible $\overline{\mathbb{F}}_p$-reps of $\text{GL}_2(k)$. \textit{"Serre conjectures"}

There are various descriptions of $W(p)$, known by the work of a number of people to be equivalent.

Spell this out carefully for $K = \mathbb{Q}_p$.

- Irreducible $\overline{\mathbb{F}}_p$-reps of $\text{GL}_2(\mathbb{F}_p)$ are

$$\sigma_{s, t} := \text{det}^t \otimes \text{Sym}^s \mathbb{F}_p^2$$

$0 \leq s \leq p-1$

$te \mathbb{Z}/(p-1)\mathbb{Z}$.
Before describing 2-dim reps of $G_{a_p}$, need to say something about the 1-dim reps.

There is a subgroup $I_p < G_{a_p}$, the inertia group, such that

$$G_{a_p}/I_p \cong \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$$

contains $x \mapsto x^p$, which generates a dense subgroup.

We say a representation of $G_{a_p}$ is unramified if it is trivial on $I_p$. 
**Def:** Set $\pi = (-p)^{\frac{1}{p-1}}$.

Let $\omega: \mathbb{G}_{a, p} \rightarrow \mathbb{F}_p^\times$ given by

$$
g \mapsto \frac{g\pi}{\pi} \pmod{p}.
$$

**Exercise:** Any mod $p$ character of $\mathbb{G}_{a, p}$ has the form $\chi \cdot \omega^a$, $a \in \mathbb{Z} \setminus (p-1)\mathbb{Z}$, $\chi$ unramified.

(Use that $\mathbb{G}_{a, p} \cong \mathbb{F}_p^\times$)

**Def:** Set $\pi_2 = (-p)^{\frac{1}{p^2-1}}$.

Let $\omega_2: \mathbb{G}_{a, p} \rightarrow \mathbb{F}_{p^2}^\times$.
\[ g \mapsto g^{\pi_2} / \pi_2 \in \mathbb{F}_{p^2}^* \]

\textbf{Prop.:} Any 2-dim. \( \mathbb{F}_p \)-rep of \( \mathbb{G}_o \) is either reducible, or else

\[ \rho \cong \chi \otimes \text{Ind}_{\mathbb{G}_q}^{\mathbb{G}_o} \omega_2^b \]

\[ (b \in \mathbb{Z}/(p^2-1)\mathbb{Z}) \]

\[ \Rightarrow \rho \mid_{I_p} \cong \omega_2^b \otimes \omega_2^{pb}. \]
Explicit def. of $W(\bar{\rho})$:

- $\sigma_{s,0} \in W(\bar{\rho}) \iff \sigma_{s,t} \in W(\bar{\rho} \otimes \omega_t)$.
- $\bar{\rho}$ irreducible:

\[ \sigma_{s,0} \in W(\bar{\rho}) \iff \bar{\rho} \mid_{\mathcal{I}_p} \omega_{s+1}^p \otimes \omega_2 \]

(Exercise: $\Rightarrow \sigma_{p-1-s, s} \in W(\bar{\rho})$)

- $\bar{\rho}$ reducible:

\[ \sigma_{s,0} \in W(\bar{\rho}) \iff \bar{\rho} \mid_{\mathcal{I}_p} (\omega_{s+1}^p \ast) \]

except
if $\rho \sim (0, x^*)$ then
$\operatorname{Ext}^1(\mathcal{O}, \omega X) \cong L \ni c$
2-dim'l canonical line
and if $x \not\in L$ then
$\sigma_{p-1,0} \in W(\rho)$ but not $\sigma_{0,0}$.

Example:

\[
\begin{pmatrix}
\omega^{s+n} & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & \omega^{s+n}
\end{pmatrix}
\Rightarrow \omega^{s+n} \otimes \begin{pmatrix}
\omega^{p-2-s} & 0 \\
0 & 1
\end{pmatrix}
\Rightarrow \sigma_{p-3-s, s+n} \in W(\rho) \text{ as well.}
\]
**Thm:** (Khare-Wintenberger, Kisin, Edixhoven, Ash-Stevens, Ribet, Gross, Coleman-Voloch, Buzzard, Wiese...) An irreducible $\overline{f} : \text{Gal}_Q \to \text{Gal}_2(\overline{\mathbb{F}_p})$ is modular of weight $k$ and level prime to $p$ if
\[ \text{TH}(\text{Sym}^{k-2}(\overline{\mathbb{F}_p})) \cap W(\overline{\text{Fl}_{\overline{a_0}}}) \neq \emptyset \]

**Thm (CEGS):** Suppose $p > 2$. The components of $\mathcal{X}_2$ are in bijection with Serre weights $\mathcal{X}(k) \leftrightarrow \sigma$ such that $\overline{\rho} \in \mathcal{X}(6) \iff \sigma \in W(\overline{\rho})$
except that for $\sigma = \chi \otimes \text{St}$,

$$\rho \in \pi(\chi \otimes \text{St}) \cup \pi(\chi) \iff \sigma \in W(\rho).$$

**Pictures:**

\[ \sigma_{p-1,0,0} \quad \sigma_{5,0} \quad \sigma_{p-1,0,0} \]

\[ w^{s+1} \text{Ind } w^s \]
meet at $L$.

Generally:

$(\chi \omega * \chi) 
* \in L$

$\sigma_{s-1,0}$

$\sigma_{0,0}$

Generally:

$(\chi_1 \omega * \chi_2) 
\chi_1 \neq \chi_2$