Moduli of Galois Representations
(CTNT 2020)

<u>Notation:</u>

residue field k

p a prime

$K/\mathbb{Q}_p$ a finite extension

$G_K := \mathrm{Gal}(\overline{K}/K)$, the absolute Galois gp. of K.

<u>Theorem:</u> (Emerton-Gee, 2019)

For each $d \geq 1$, there exists a finite type algebraic stack $\mathcal{X}_d$ over $\mathbb{F}_p$ with the following properties:

(1) For each finite $\mathbb{F}/\mathbb{F}_p$

$\mathbb{F}_p$-pts of $\mathcal{X}_d$ $\longleftrightarrow$ continuous reps $\bar{\rho}: G_K \to GL_d(\mathbb{F})$

(2) $\overline{\mathcal{X}}_d$ is equidimensional of dim. $[K:\mathbb{Q}_p]\binom{d}{2}$.

(3) A description of the irreducible components of $\overline{\mathcal{X}}_d$, namely:

each component $\mathcal{X}(\sigma) \subseteq \overline{\mathcal{X}}_d$ has a dense open subset $U \subseteq \mathcal{X}(\sigma)$ whose $\mathbb{F}$-pts. are certain specified

Successive ext$^{\underline{ns}}$s of characters.

Application: (Emerton-Gee)
Every $\bar{\rho}$ as in (1) lifts to characteristic 0.
(Previously known for $d \leq 3$.)

Today (joint with Caraiani, Emerton, Gee): a more precise description of the components of $\overline{\mathcal{X}}_2$ $(p>2)$.

To each $\bar{\rho}: G_K \to GL_2(\overline{\mathbb{F}}_p)$ there is an associated set

$W(\bar{\rho})$ of irreducible $\overline{\mathbb{F}}_p$-reps of $GL_2(k)$. "Serre wts"

There are various descriptions of $W(\bar{\rho})$, known by the work of a number of people to be equivalent.

Spell this out carefully for $K = \mathbb{Q}_p$.

- Irreducible $\overline{\mathbb{F}}_p$-reps of $GL_2(\mathbb{F}_p)$ are

$$\sigma_{s,t} := \det{}^t \otimes \operatorname{Sym}^s \overline{\mathbb{F}}_p^2$$

$0 \leq s \leq p-1$

$t \in \mathbb{Z}/(p-1)\mathbb{Z}$.

$GL_2(\mathbb{F}_p) \curvearrowright \overline{\mathbb{F}}_p^2$

- Before describing 2-dim'l reps of $G_{\mathbb{Q}_p}$, need to say something about the 1-dim'l reps.

There is a subgroup $I_p \triangleleft G_{\mathbb{Q}_p}$, the inertia group, such that

$$G_{\mathbb{Q}_p}/I_p \cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

contains $x \mapsto x^p$, which generates a dense subgroup.

We say a representation of $G_{\mathbb{Q}_p}$ is unramified if it is trivial on I_p.

Def: Set $\pi = (-p)^{\frac{1}{p-1}}$,
Let
$$\omega : G_{\mathbb{Q}_p} \longrightarrow \mathbb{F}_p^{\times} \quad \text{given by}$$
$$g \longmapsto \frac{g\pi}{\pi} \pmod{p}.$$

Exercise: Any mod p character of $G_{\mathbb{Q}_p}$ has the form $\chi \cdot \omega^a$ $\quad a \in \mathbb{Z}/(p-1)\mathbb{Z}$
χ unramified.

(Use that $G_{\mathbb{Q}_p}^{ab} \simeq \widehat{\mathbb{Q}_p^{\times}}$)

Def: Set $\pi_2 = (-p)^{\frac{1}{p^2-1}}$.

Let $\omega_2 : G_{\mathbb{Q}_p} \longrightarrow \mathbb{F}_{p^2}^{\times}$

$$g \mapsto g^{\pi_2}/\pi_2 \in \mathbb{F}_{p^2}^\times$$

<u>Not</u> a char. of $G_{\mathbb{Q}_p}$, but it <u>is</u> a character of $G_{\mathbb{Q}_{p^2}}$, hence of I_p.

<u>Prop</u>: Any 2-dim'l $\overline{\mathbb{F}}_p$-rep of $G_{\mathbb{Q}_p}$ is either reducible, or else

$$\bar{\rho} \cong \chi \otimes \mathrm{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p} \omega_2^b$$

$$(b \in \mathbb{Z}/(p^2-1)\mathbb{Z})$$

$$\Rightarrow \bar{\rho}|_{I_p} \cong \omega_2^b \oplus \omega_2^{pb}.$$

Explicit def. of $W(\bar{\rho})$:

- $\sigma_{s,0} \in W(\bar{\rho}) \iff \sigma_{s,t} \in W(\bar{\rho} \otimes \omega^t)$.

- $\bar{\rho}$ irreducible:

$\sigma_{s,0} \in W(\bar{\rho})$

$$\iff \quad \bar{\rho}|_{I_p} \cong \omega_2^{s+1} \oplus \omega_2^{p(s+1)}.$$

(<u>Exercise</u>: $\Rightarrow \sigma_{p-1-s,s} \in W(\bar{\rho})$.)

- $\bar{\rho}$ reducible:

$\sigma_{s,0} \in W(\bar{\rho})$

$$\iff \quad \bar{\rho}|_{I_p} \sim \begin{pmatrix} \omega^{s+1} & * \\ 0 & 1 \end{pmatrix}$$

<u>except</u>

$\rightarrow s = 0, p-1$

if $\bar{\rho} \sim \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ then

$\underset{\substack{\uparrow \\ \text{2-dim'l}}}{\mathrm{Ext}^1(\chi, \omega\chi)} \supseteq \underset{\substack{\uparrow \\ \text{canonical line}}}{L}$

and if $* \notin L$ then

$\sigma_{p-1,0} \in W(\bar{\rho})$ but <u>not</u> $\sigma_{0,0}$.

<u>Example</u>:

$$\begin{pmatrix} \omega^{s+1} & 0 \\ 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & \omega^{s+1} \end{pmatrix}$$

$$\simeq \omega^{s+1} \otimes \begin{pmatrix} \omega^{p-2-s} & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \sigma_{p-3-s,\, s+1} \in W(\bar{\rho})$ as well.

Thm: (Khare-Wintenberger, Kisin, Edixhoven, Ash-Stevens, Ribet, Gross, Coleman-Voloch, Buzzard, Wiese....)

An irreducible $\bar{r} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is modular of weight k and level prime to p iff

$$JH(\mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p) \cap W(\bar{r}|_{G_{\mathbb{Q}_p}}) \neq \emptyset$$

Thm (CEGS) Suppose $p > 2$. The components of $\overline{\mathcal{X}}_2$ are in bijection with Serre weights

$$\mathcal{X}(\sigma) \Longleftrightarrow \sigma$$

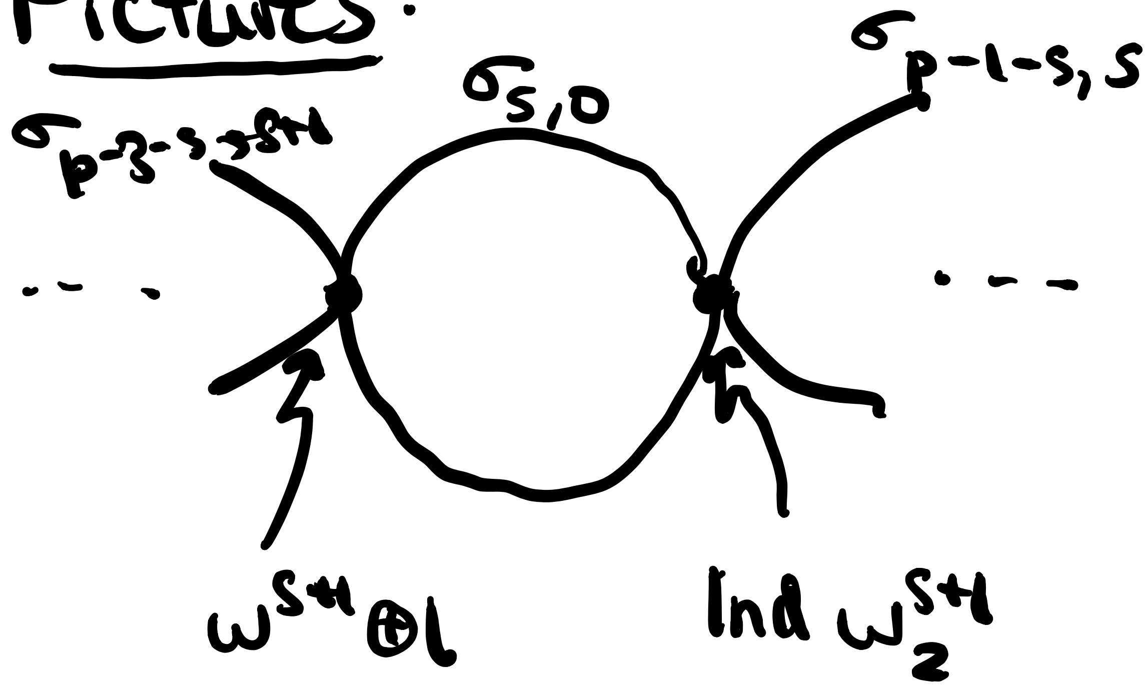
such that

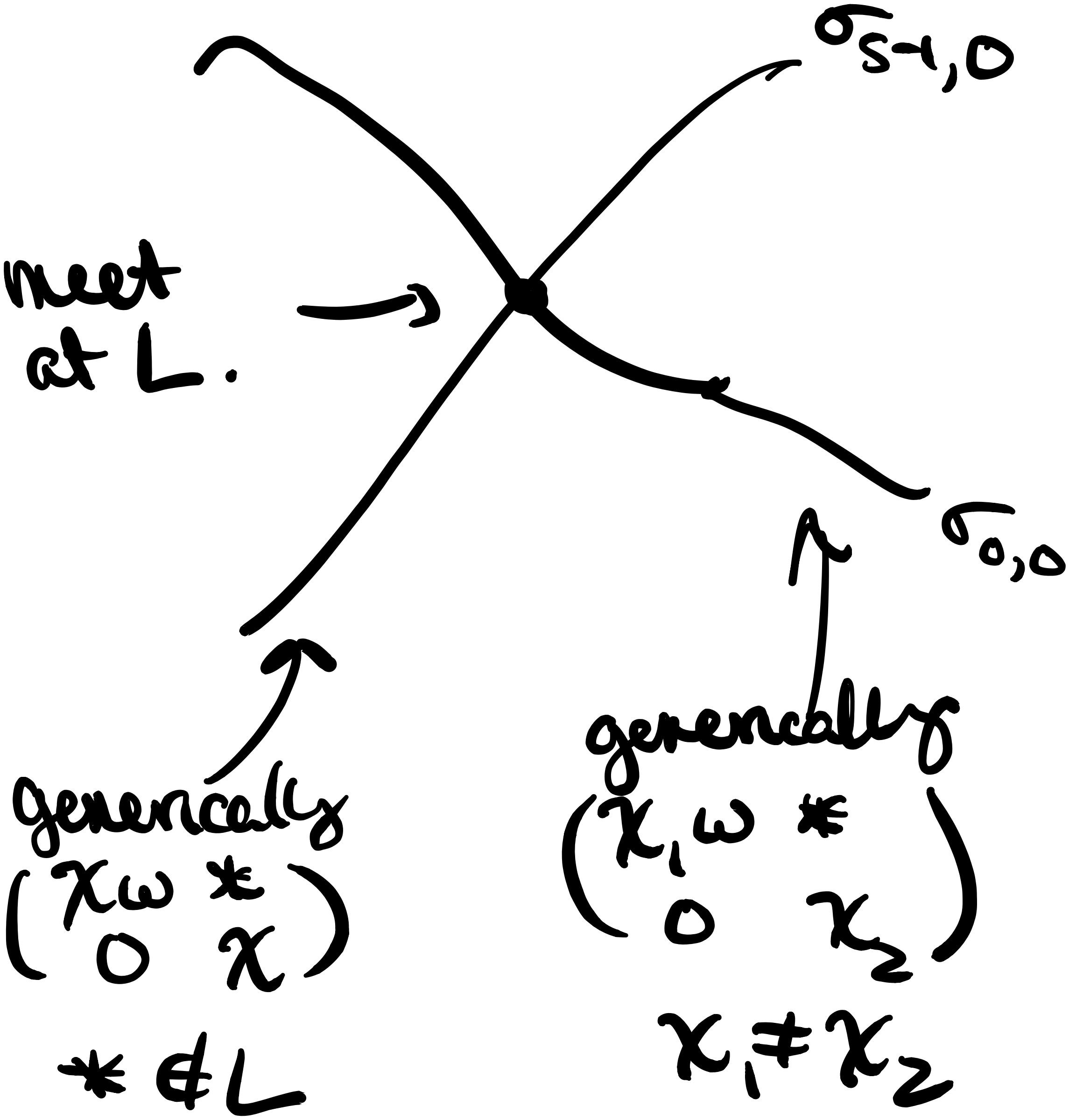
$$\bar{\rho} \in \mathcal{X}(\sigma) \Longleftrightarrow \sigma \in W(\bar{\rho})$$

except that for $\sigma = \chi \otimes \mathrm{St}$,

$$\bar{\rho} \in \mathcal{X}(\chi \otimes \mathrm{St}) \cup \mathcal{X}(\chi) \iff \sigma \in W(\bar{\rho}).$$

<u>Pictures</u>:




$\sigma_{s-1,0}$
$\sigma_{0,0}$
meet at L.
generically $\begin{pmatrix} \chi\omega & * \\ 0 & \chi \end{pmatrix}$
$* \notin L$
generically $\begin{pmatrix} \chi_1\omega & * \\ 0 & \chi_2 \end{pmatrix}$
$\chi_1 \neq \chi_2$