The Mahler measure of a genus 3 family

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Extending Mersenne's construction

Pierce (1918) : A construction for finding large prime numbers.

$P(x) \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - r_i)$$

$$\Delta_n = \prod_i (r_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1.$$

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What are the best polynomials?

D. H. Lehmer (1933) : To improve the chances of finding a big prime, we need Δ_n that grows slowly.

$$\lim_{n\to\infty} \frac{|\Delta_{n+1}|}{|\Delta_n|} > 1, \text{ but close to } 1.$$

$$\lim_{n\to\infty} \frac{|r^{n+1}-1|}{|r^n-1|} = \begin{cases} |r| & \text{if } |r| > 1, \\ 1 & \text{if } |r| < 1. \end{cases}$$

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Mahler measure

For

$$P(x) = a\prod_i (x - r_i)$$

$$M(P) = |a| \prod_{|r_i|>1} |r_i|, \qquad \mathrm{m}(P) = \log|a| + \sum_{|r_i|>1} \log|r_i|.$$

Thus, we want,

$$M(P) > 1 \text{ but close, or } \mathrm{m}(P) > 0 \text{ but close.}$$

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Kronecker (1857) : $P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathrm{m}(P) = 0 \iff P(x) \text{ is product of cyclotomic and monomials.}$$

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Lehmer's Question

Lehmer (1933)

Given $\varepsilon > 0$*, can we find a polynomial* $P(x) \in \mathbb{Z}[x]$ *such that* $0 < \mathrm{m}(P) < \varepsilon$*?*

Conjecture: No.

$$\mathrm{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 0.162357612\ldots$$

Conjecture: This polynomial is the best possible.

With this polynomial, Lehmer found

$$\sqrt{\Delta_{379}} = 1,794,327,140,357.$$

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Mahler measure of multivariable polynomials

$P \in \mathbb{C}(x_1, \dots, x_n)^\times$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} \mathrm{m}(P) &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}. \end{aligned}$$

Jensen's formula implies

$$\mathrm{m}(P) = \log |a| + \sum_{|r_i|>1} \log |r_i| \quad \text{for} \quad P(x) = a \prod_i (x - r_i)$$

$$M(P) := \exp(\mathrm{m}(P)).$$

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Boyd–Lawton Theorem

Boyd (1981), Lawton (1983)

For $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathrm{m}(P(x_1, \ldots, x_n))$$

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The Mahler measure of several variable polynomials does not say much new about Lehmer's Question.

Mahler measure is ubiquitous!

- Interesting questions about distribution of values
- Heights
- Volumes in hyperbolic space
- Entropy of certain arithmetic dynamical systems
- Special values of L-functions

Examples in several variables

Smyth (1981)

- $$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$
- $$\mathrm{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

L. (2006)

- $$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5)$$
- Known formulas for
$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

The measures of a family of elliptic curves

$$m(k) := \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

$$X = -\frac{1}{xy}, \qquad Y = \frac{(y-x)(1+xy)}{2x^2y^2}$$

$$E_k : Y^2 = X^3 + (k^2/4 - 2)X^2 + X$$

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Conjecture (Boyd (1998))

$$m(k) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

$s_k \in \mathbb{Q}$ *of low height (often in* $\mathbb{Z}$*)*

Boyd's conjectures

$$m(k) = \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k}$$

k	s_k	N_k	k	s_k	N_k
1	*1*	*15*	11	-8	1155
2	*1*	*24*	*12*	*1/2*	*48*
3	*1/2*	*21*	13	-4	663
4	*	*	14	8	840
5	*1/6*	*15*	15	-24	3135
6	2	120	*16*	*1/11*	*15*
7	2	231	17	-24	4641
8	*1/4*	*24*	18	-16	1848
9	2	195	19	-40	6555
10	-8	840	20	2	240

Rodriguez-Villegas (1997), Rogers & Zudilin (2010, 2011), Mellit (2011), Zudilin (2014), Brunault (2015), Brunault (2016), L., Samart, and Zudilin (2016)

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Why do we get nice numbers?

In many cases, the Mahler measure is a special period coming from Beĭlinson's conjectures!

Deninger (1997) , Rodriguez-Villegas (1997) , Boyd (1998).

The regulator

$$\mathrm{m}(P) = \frac{1}{2\pi} \int_{\gamma_P} \eta(x, y),$$

where

$$\eta(x, y) = \log|y| d \arg x - \log|x| d \arg y$$

and

$$d \arg x = \mathrm{Im}\left(\frac{dx}{x}\right).$$

$\eta(x, y)$ is a closed differential form defined on $P = 0$ minus the set S of zeros and poles of x, y.

$$\int_\gamma \eta(x, y) \overset{\text{Beĭlinson}}{\longrightarrow} L'(E, 0)$$

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Relationships between Mahler measures

Idea: we have two polynomials P and Q conjectured to be related to the same $L'(E,0)$.

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Can we prove a relationship between

$$\int_{\gamma_P} \eta(x,y) \text{ and } \int_{\gamma_Q} \eta(x,y)?$$

This gives a relationship between

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Sometimes $\int_{\gamma_P} \eta(x,y)$ can be written as a function on the divisors of x and y in the curve given by $P=0$. A relationship between P and Q may translate into a relationship between the regulators.

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Relationships - some examples

- Rodriguez-Villegas (2000)

$$7\mathrm{m}(y^2+2xy+y-x^3-2x^2-x)=5\mathrm{m}(y^2+4xy+y-x^3+x^2).$$

- L. & Rogers (2007), L. (2010)

$$m(8)=4m(2), \quad m(5)=6m(1).$$

- Identities involving genus 2 curves conjectured by Boyd, proven by Bertin and Zudilin (2016-17).

Genus 3 and Genus 1

Liu and Qin (2019+) studied many new numerical examples involving genus 2 and genus 3 curves including

$$P_k(x,y) = y^2 + (x^6 + kx^5 - x^4 + (2-2k)x^3 - x^2 + kx + 1)y + x^6$$

and

$$Q_k(x,y) = xy^2 + (kx-1)y - x^2 + x.$$

Theorem (L. & Wu, 2020+)

For $k \geq 2$,

$$\mathrm{m}(Q_k) = \mathrm{m}(P_k).$$

Genus 3 and Genus 1 - Idea

$Q_k = 0$ has genus 1 and is birational to

$$E_k : Y^2 = X^3 + (k^2 - 4)X^2 - 8kX + 16$$

$C_k : P_k = 0$ has genus 3 if $k \neq \pm 2$. It has an involution $\sigma : (x, y) \to (1/x, 1/y)$. Then

$$C_k/\sigma \cong E_k.$$

k	2	3	4	5	6	7	8	9	10
s_k	$-1/2$	-1	-2	-4	6	14	-18	36	52
N_k	37	79	197	469	997	1907	3349	5497	8549

Table: Numerical values of $1/s_k$ for $\mathrm{m}(Q_k) = \mathrm{m}(P_k) = \frac{L'(E_k,0)}{s_k}$.

In conclusion...

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- Three variable examples with $L'(E, -1)$?

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- The regulator yields some insight about nontrivial identities between Mahler measures.
- Exact formulas with $g > 1$?
- Three variable examples with $L'(E, -1)$?

Thanks for your attention!

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