Class Groups and (Fine) Selmer Groups in Iwasawa Theory

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IDEA: Studying class groups or Selmer groups in isolation is hard. But, these properties stabilize in *certain* towers; in such cases studying them becomes easier.

Classical Iwasawa Theory

Consider the tower

$$\mathbb{Q} = \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \cdots \subset \mathbb{Q}_n \subset \cdots \subset \bigcup_n \mathbb{Q}_n =: \mathbb{Q}_{cyc}$$

where $\mathbb{Q}_n$ is the unique subfield of $\mathbb{Q}(\mu_{p^{n+1}})$ such that $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

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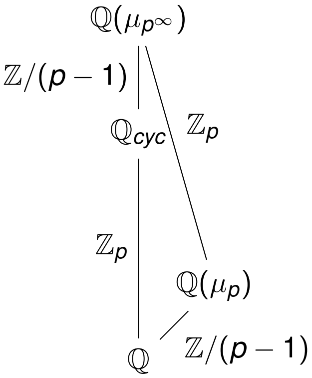
For a number field $F/\mathbb{Q}$, the cyclotomic $\mathbb{Z}_p$ extension always exists but it may not be the only $\mathbb{Z}_p$-extension.

Field Diagram



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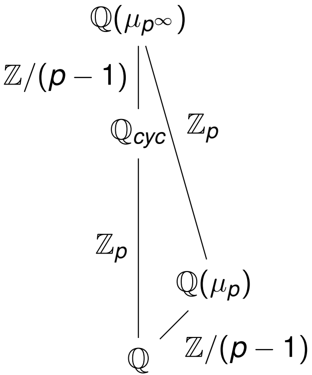

We have

$$\mathrm{Gal}\left(\mathbb{Q}\left(\mu_{p^\infty}\right)/\mathbb{Q}\right) \simeq \mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{Z}/(p-1) \simeq (1+p\mathbb{Z}_p) \times \mathbb{Z}/(p-1).$$

Iwasawa algebra

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Definition

The ***Iwasawa algebra****,* $\Lambda(G) := \varprojlim \mathbb{Z}_p[G/H]$
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$$\Lambda(\Gamma) = \mathbb{Z}_p[\![\Gamma]\!] \xrightarrow{\sim} \mathbb{Z}_p[\![T]\!]$$
$$\gamma \mapsto 1 + T$$

Structure Theorem: Iwasawa and Serre

Theorem

Let M be a finitely generated $\Lambda(\Gamma)$-module. Then

$$M \sim \Lambda(\Gamma)^r \oplus \left(\bigoplus_{i=1}^{t} \Lambda(\Gamma)/p^{n_i} \right) \oplus \left(\bigoplus_{j=1}^{s} \Lambda(\Gamma)/f_j^{m_j} \right)$$

where f_j are irreducible distinguished polynomials in $\mathbb{Z}_p[T]$.

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$$\lambda(M) := \sum_{j=1}^{s} m_j \deg(f_j).$$

Classical Theorem

Theorem (Iwasawa)

Let F_∞/F be a $\mathbb{Z}_p$-extension and let e_n be the integer so that $p^{e_n}||h_n$ where h_n is the order of the class group of F_n. There exist integers $\lambda,\ \mu \geq 0$ and ν such that

$$e_n = \lambda n + \mu p^n + \nu$$

for all n sufficiently large where $\lambda,\ \mu,\ \nu$ are all independent of n.

Classical $\mu = 0$ Conjecture

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This is known only when $F/\mathbb{Q}$ is an Abelian extension. This was proved by Ferrero-Washington (1979).

Selmer Groups of Elliptic Curves

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The p**-primary Selmer group** of E/L for a finite Galois extension L/F contained in F_S is given by the exact sequence

$$0 \to \mathrm{Sel}(E/L) \to H^1\left(\mathrm{Gal}\left(F_S/L\right),\ E[p^\infty]\right) \to \bigoplus_{v \in S} \bigoplus_{w|v} H^1\left(L_w,\ E\right)[p^\infty].$$

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We have,

$$\mathrm{Sel}\left(E/F_\infty\right) = \varinjlim_{L} \mathrm{Sel}\left(E/L\right)$$

where L runs over all finite extensions of F contained in the $\mathbb{Z}_p$-extension F_∞/F.

Iwasawa Theory of Elliptic Curves: The Work of Mazur

When the Galois module M is a discrete p-primary Abelian group or a compact pro-p Abelian group, its **Pontryagin dual** is defined as

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Theorem (Mazur)

The Pontryagin dual of $\mathrm{Sel}\left(E/F_{cyc}\right)$*, denoted by* $X\left(E/F_{cyc}\right)$*, is a finitely generated* $\Lambda(\Gamma)$*-module.*

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If $X\left(E/F_{cyc}\right)$ is $\Lambda(\Gamma)$-torsion, the Structure Theorem holds as in the classical case. There are examples (over $\mathbb{Q}$) where $\mu(X) > 0$.

Coates-Sujatha Conjecture A

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Conjecture (Coates-Sujatha (2005))

Let p be any odd prime and $Y(E/F_{cyc})$ be the Pontryagin dual of the fine Selmer group. Then,
$Y(E/F_{cyc})$ is $\Lambda(\Gamma)$-torsion and $\mu(Y) = 0$.

$Y(E/F_{cyc})$ is expected to be $\Lambda(\Gamma)$-torsion independent of the reduction type at p.

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$Y(E/F_{cyc})$ is expected to be $\Lambda(\Gamma)$-torsion independent of the reduction type at p. This is the elliptic curve analogue of the *weak Leopoldt conjecture*. It is known for elliptic curves over $\mathbb{Q}$ by a deep result of Kato.

Fine Selmer Group

We define

$$R(E/L) := \ker\left(H^1\left(\mathrm{Gal}\left(F_S/L\right), E[p^\infty]\right) \to \bigoplus_{v \in S} \bigoplus_{w|v} H^1\left(L_w,\ E[p^\infty]\right)\right).$$

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Taking direct limits as before, define

$$R(E/F_\infty) := \varinjlim_{L} R(E/L)$$

where L runs over all finite extensions of F contained in F_∞.

Evidence for Conjecture A

Theorem (K.-Sujatha)

Let F be a number field and E be an elliptic curve of rank 0 over F. Assume that the Shafarevich-Tate group of the elliptic curve E over F is finite. Varying over all primes of good ordinary reduction, $\mathrm{Sel}(E/F_{cyc})$ is trivial outside a set of density 0. In particular, Conjecture A holds for $Y(E/F_{cyc})$.

Relating the Two Conjectures

Briefly recall the two conjectures discussed so far

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In a cyclotomic $\mathbb{Z}_p$-extension, $\mu = 0$.

Conjecture (Coates-Sujatha (2005))

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Let $F/\mathbb{Q}$ be a number field and $p \neq 2$ such that $E[p] \subset E(F)$. Then Conjecture A for $Y(E/F_{cyc})$ is equivalent to the classical $\mu = 0$ Conjecture.

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It is natural to ask the following question:

Question

Are (fine) Selmer groups closely related to the class group in other p-adic Lie extensions (in particular, other $\mathbb{Z}_p$ extensions)?

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- non-Abelian non p-adic analytic extensions
- quadratic extensions
- cyclic $\mathbb{Z}/p\mathbb{Z}$ extensions

Non-Cyclotomic $\mathbb{Z}_p$-Extensions

Theorem (Iwasawa)

Let F be the cyclotomic field of p-th or 4-th roots of unity according as $p > 2$ or $p = 2$. For any given integer $N \geq 1$, there exists a cyclic extension L/F of degree p and a $\mathbb{Z}_p$-extension L_∞/L such that

$$\mu \geq N.$$

Theorem (K. 2020)

Let F be the cyclotomic field of p-th roots of unity for $p > 2$. Let E/F be an elliptic curve such that $E(F)[p] \neq 0$. Suppose the analogue of the Weak Leopoldt Conjecture holds. Given an integer $N \geq 1$, there exists a cyclic extension L/F of degree p and a $\mathbb{Z}_p$-extension L_∞/L such that

$$\mu(Y(E/L_\infty)) \geq N.$$

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- Let F_+ be the maximal totally real subfield of F. Construct a (non-)cyclotomic $\mathbb{Z}_p$-extension F_∞/F which is Galois over F_+.

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- Choose prime ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_t$, $t \geq 1$, in F which are prime to p and are totally split in F_∞/F. Let η be a non-zero element of F which is divisible exactly by the first power of $\mathfrak{l}_i$ for $1 \leq i \leq t$. Set $L = F\left(\sqrt[p]{\eta}\right)$ and $L_\infty = LF_\infty$.

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- Relate the (p-rank of) fine Selmer groups to the (p-rank of) class groups.
- Finally using Iwasawa's result show

$$\mu\left(Y\left(E/L_\infty\right)\right) \geq c_1 t + c_2(E/F).$$

More General Results

In pro-*p* *p*-adic Lie extensions, there is now a general strategy to show that the μ-invariant of (fine) Selmer groups can be arbitrarily large in extensions where it is known that the μ-invariant associated to the class group is arbitrarily large.

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More precisely, using the work of Hajir-Maire (2019), it is possible to show that there are nilpotent, uniform, pro-p, p-adic Lie extensions of number fields with arbitrarily large μ-invariant of (fine) Selmer groups.

Thank you!!