Stacky Curves in Characteristic p

Andrew J. Kobin

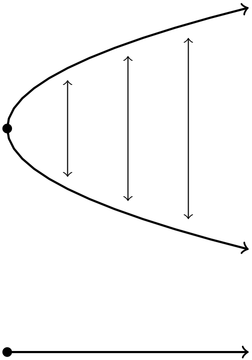
ak5ah@virginia.edu

Connecticut Summer School in Number Theory

June 12, 2020

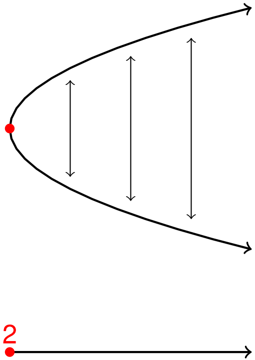
Introduction

Common problem: all sorts of information is lost when we consider quotient objects and/or singular objects.



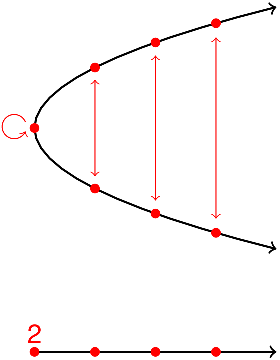
Introduction

Solution: Keep track of lost information using *orbifolds* (topological and intuitive) or *stacks* (algebraic and fancy).



Introduction

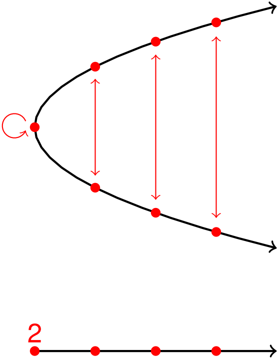
Solution: Keep track of lost information using *orbifolds* (topological and intuitive) or *stacks* (algebraic and fancy).



Example: For the plane curve $X : y^2 - x = 0$, stacks remember automorphisms like $(x, y) \leftrightarrow (x, -y)$ using groupoids

Introduction

Solution: Keep track of lost information using *orbifolds* (topological and intuitive) or *stacks* (algebraic and fancy).

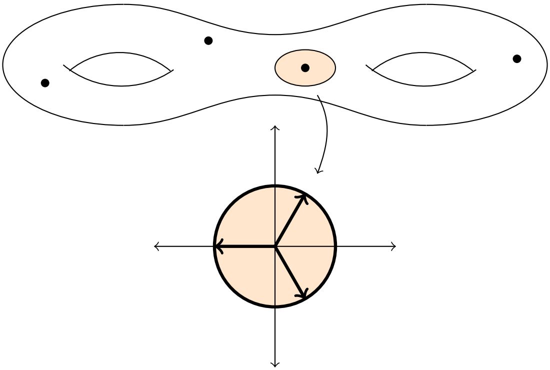


Goal: Classify stacky curves (= orbifold curves) in char. p (preprint available!)

Complex Orbifolds

Definition

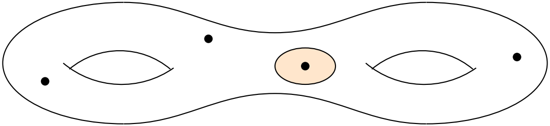
A **complex orbifold** is a topological space admitting an atlas $\{U_i\}$ where each $U_i \cong \mathbb{C}^n/G_i$ for a finite group G_i, satisfying compatibility conditions (think: manifold atlas but with extra info).



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

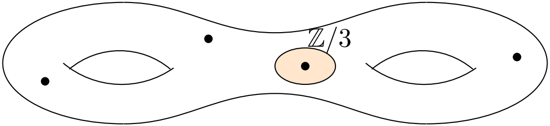
One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

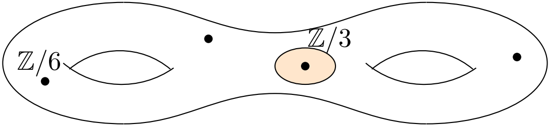
One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

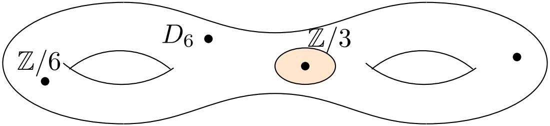
One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

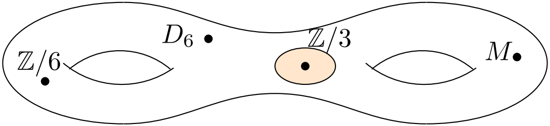
One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

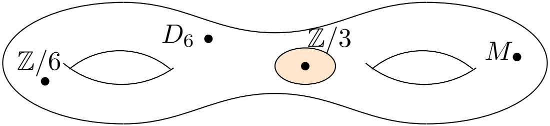
One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.

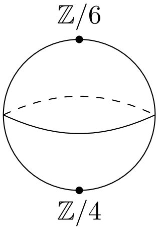


Focus on curves for the rest of the talk

An Example

Example

The (compactifed) moduli space of complex elliptic curves is a stacky $\mathbb{P}^1$ with a generic $\mathbb{Z}/2$ and a special $\mathbb{Z}/4$ and $\mathbb{Z}/6$.



Consequence: can deduce dimension formulas for modular forms from Riemann–Roch formula for stacks.

Goal: Classify stacky curves in char. p.

Main obstacle to overcome:

- In char. 0, local structure is determined by a cyclic group action.
- In char. p, this is not enough information – need more invariants than just the order of a cyclic group.

Results (K. '20):

- Every p-cover of curves factors étale-locally through an **Artin–Schreier root stack.**
- Every stacky curve with order p automorphism group is étale-locally an Artin–Schreier root stack.
- For any algebraic curve X, there are infinitely many non-isomorphic Deligne–Mumford stacks with coarse space X and degree p automorphism groups at the same sets of points.

Root Stacks

Key fact: in char. 0, all stabilizers (automorphism groups) are *cyclic*.

So stacky curves can be locally modeled by a *root stack*: charts look like

$$U \cong [\operatorname{Spec} A/\mu_n]$$

where $A = K[y]/(y^n - \alpha)$ and μ_n is the group of nth roots of unity.

(Think: degree n branched cover mod μ_n-action, but remember the action using groupoids.)

Root Stacks

More rigorously:

Definition (Cadman '07, Abramovich–Olsson–Vistoli '08)

Let X be a scheme and $L \to X$ a line bundle with section $s : X \to L$. The **nth root stack** of X along (L, s) is the fibre product

$$\begin{tikzcd} \sqrt[n]{(L,s)/X} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] & x \arrow[d, mapsto] \\ X \arrow[r, "{(L,s)}"] & {[\mathbb{A}^1/\mathbb{G}_m]} & x^n \end{tikzcd}$$

Here, $[\mathbb{A}^1/\mathbb{G}_m]$ is the classifying stack for pairs (L, s).

Interpretation: $\sqrt[n]{(L,s)/X}$ admits a canonical tensor nth root of (L, s), i.e. (M, t) such that $M^{\otimes n} = L$ and $t^n = s$ (after pullback).

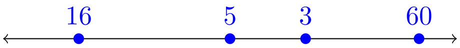
Root Stacks

Theorem (Geraschenko–Satriano '15)

Every smooth separated **tame** *Deligne–Mumford stack of finite type with trivial generic stabilizer is** *a root stack over its coarse space.*

Corollary

Tame *stacky curves are completely described by their coarse space and a finite list of numbers corresponding to the orders of cyclic stabilizers at a finite number of stacky points.*



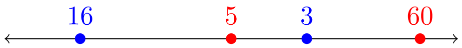
Root Stacks

Theorem (Geraschenko–Satriano '15)

Every smooth separated **tame** *Deligne–Mumford stack of finite type with trivial generic stabilizer is* a root stack over its coarse space.*

Corollary

Tame *stacky curves are completely described by their coarse space and a finite list of numbers corresponding to the orders of cyclic stabilizers at a finite number of stacky points.*



What happens with **wild** stacky curves in char. p?

Artin–Schreier Root Stacks

In trying to classify **wild** stacky curves in char. p, we face the following problems:

1. Stabilizer groups need not be cyclic (or even abelian)
2. Cyclic $\mathbb{Z}/p^n\mathbb{Z}$-covers of curves occur in families
3. Root stacks don't work
   - Finding $M^{\otimes p}$ is a problem
   - $[\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m], x \mapsto x^p$ is a problem

Key case: cyclic $\mathbb{Z}/p\mathbb{Z}$ stabilizers

Artin–Schreier Root Stacks

Idea: replace **tame** cyclic covers $y^n = f(x)$ with **wild** cyclic covers $y^p - y = f(x)$.

More specifically: **Artin–Schreier theory** classifies cyclic degree p-covers of curves in terms of the **ramification jump** (e.g. if $f(x) = x^m$ then m is the jump).

This suggests introducing wild stacky structure using the local model

$$U = [\operatorname{Spec} A/(\mathbb{Z}/p)]$$

where $A = k[y]/(y^p - y - f(x))$ and $\mathbb{Z}/p$ acts additively.

Artin–Schreier Root Stacks

How do we do it?

$$\begin{tikzcd} \sqrt[n]{(L,s)/X} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\ X \arrow[r, "{(L,s)}"] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, mapsto] \\ x^n \end{tikzcd}$$

Artin–Schreier Root Stacks

How do we do it?

$$\begin{tikzcd} \sqrt[n]{(L,s)/X} \arrow[r] \arrow[d] & {[\mathbb{P}^1/\mathbb{G}_a]} \arrow[d] \\ X \arrow[r, "{(L,s)}"] & {[\mathbb{P}^1/\mathbb{G}_a]} \end{tikzcd} \qquad \begin{tikzcd} {[u,v]} \arrow[d, mapsto] \\ {[u^p, v^p - vu^{p-1}]} \end{tikzcd}$$

Artin–Schreier Root Stacks

How do we do it?

$$\begin{tikzcd}
\sqrt[n]{(L,s)/X} \arrow[r] \arrow[d] & {[\mathbb{P}^1/\mathbb{G}_a]} \arrow[d] \\
X \arrow[r, "{(L,s,f)}"] & {[\mathbb{P}^1/\mathbb{G}_a]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[u,v]} \arrow[d, mapsto] \\
{[u^p, v^p - vu^{p-1}]}
\end{tikzcd}$$

Artin–Schreier Root Stacks

How do we do it?

$$\begin{tikzcd} \wp_1^{-1}((L,s,f)/X) \arrow[r] \arrow[d] & {[\mathbb{P}^1/\mathbb{G}_a]} \arrow[d] \\ X \arrow[r, "{(L,s,f)}"] & {[\mathbb{P}^1/\mathbb{G}_a]} \end{tikzcd} \qquad \begin{tikzcd} {[u,v]} \arrow[d, mapsto] \\ {[u^p, v^p - vu^{p-1}]} \end{tikzcd}$$

Artin–Schreier Root Stacks

Definition (K.)

Fix $m \geq 1$. Let X be a scheme, $L \to X$ a line bundle and $s : X \to L$ and $f : X \to L^{\otimes m}$ two sections not vanishing simultaneously. The **Artin–Schreier root stack** of X with jump m along (L, s, f) is the normalized pullback

$$\begin{tikzcd} \wp_m^{-1}((L,s,f)/X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nu\lrcorner"] & {[\mathbb{P}(1,m)/\mathbb{G}_a]} \arrow[d] \\ X \arrow[r, "{(L,s,f)}"] & {[\mathbb{P}(1,m)/\mathbb{G}_a]} \end{tikzcd} \qquad \begin{tikzcd} {[u,v]} \arrow[d, mapsto] \\ {[u^p, v^p - vu^{m(p-1)}]} \end{tikzcd}$$

where

- $\mathbb{P}(1, m)$ is the weighted projective line with weights $(1, m)$
- $\mathbb{G}_a = (k, +)$, acting additively
- $[\mathbb{P}(1, m)/\mathbb{G}_a]$ is the classifying stack for triples (L, s, f) up to the principal part of f.

Artin–Schreier Root Stacks

$$\begin{tikzcd}
\wp_m^{-1}((L,s,f)/X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nu\lrcorner", very near start] & {[\mathbb{P}(1,m)/\mathbb{G}_a]} \arrow[d] & {[u,v]} \arrow[d, mapsto] \\
X \arrow[r, "{(L,s,f)}"] & {[\mathbb{P}(1,m)/\mathbb{G}_a]} & {[u^p, v^p - vu^{m(p-1)}]}
\end{tikzcd}$$

Interpretation: $\wp_m^{-1}((L,s,f)/X)$ admits a canonical pth root of L, i.e. a line bundle M such that $M^{\otimes p} = L$, and an AS root of s.

Artin–Schreier Root Stacks

Key example:

Example (K.)

Consider the AS cover

$$\begin{tikzcd} Y \,:\, y^p - y = x^{-m} \arrow[d, "\mathbb{Z}/p"'] \\ \mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1] \end{tikzcd}$$

where k is an algebraically closed field of characteristic p. Then

$$\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong [Y/(\mathbb{Z}/p)].$$

In general, every AS root stack is étale-locally isomorphic to such an "elementary AS root stack".

Classification of (Some) Wild Stacky Curves

So let's classify us some wild stacky curves!
(Assume: everything defined over $k = \bar{k}$)

Theorem 1 (K. '20)

Every Galois cover of curves $\varphi : Y \to X$ with an inertia group $\mathbb{Z}/p$ factors étale-locally through an Artin–Schreier root stack:

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] & & X \\ V \arrow[u, "\text{ét}"] \arrow[r] & \wp_m^{-1}((L,s,f)/U) \arrow[r] & U \arrow[u, "\text{ét}"'] \end{tikzcd}$$

Informal consequence: there are infinitely many non-isomorphic stacky curves over $\mathbb{P}^1$ with a single stacky point of order p.

This phenomenon only occurs in char. p.

Classification of (Some) Wild Stacky Curves

Main result:

Theorem 2 (K. '20)

Every stacky curve $\mathcal{X}$ with a stacky point of order p is étale-locally isomorphic to an Artin–Schreier root stack $\wp_m^{-1}((L, s, f)/U)$ over an open subscheme U of the coarse space of $\mathcal{X}$.

This can even be done globally if $\mathcal{X}$ has coarse space $\mathbb{P}^1$:

Theorem 3 (K. '20)

If $\mathcal{X}$ has coarse space $\mathbb{P}^1$ and all stacky points of $\mathcal{X}$ have order p, then $\mathcal{X}$ is isomorphic to a fibre product of AS root stacks of the form $\wp_m^{-1}((L, s, f)/\mathbb{P}^1)$ for $(m, p) = 1$ and (L, s, f).

Generalizations

What about $\mathbb{Z}/p^2$-covers, stacky points of order p^2, and beyond?

For cyclic stabilizer groups $\mathbb{Z}/p^n$, Artin–Schreier theory is subsumed by **Artin–Schreier–Witt theory**:

- AS equations $y^p - y = f(x)$ are replaced by Witt vector equations $\underline{y}^p - \underline{y} = \underline{f}(\underline{x}) = (f_0(\underline{x}), \ldots, f_n(\underline{x}))$.
- Covers are characterized by *sequences of ramification jumps*.
- Local structure is $U = [\operatorname{Spec} A/(\mathbb{Z}/p^n)]$ where

$$A = K[\underline{y}]/(\underline{y}^p - \underline{y} - \underline{f})$$

  where $\underline{f} = (f_0, \ldots, f_{n-1})$ is a Witt vector over $\overline{K}$.

This local structure can be formally introduced using **Artin–Schreier–Witt root stacks** (work in progress).

Generalizations

Thank you!