Topology and Diophantine Equations

David Conwin (joint w/ Tomer Schlank)

§1 History

Let \( k \) n.f. \( X/k \) variety

\( \square \) How to determine if \( X(k) = \emptyset \)?

\[ X(k) = \emptyset \iff X(A_k) = \emptyset \]

Local-global principle

Lind-Reichardt (40's)

(Selmer) counterexamples. Used QR to prove \( X(k) = \emptyset \)
\[ QR \Rightarrow CFT \]

Manin (‘71) used CFT via Brauer group \( H_{et}^2(X; G_m) \)

defined \( X(k) \subseteq X(A_k)^{Br} \subseteq X(A_k) \)

can be empty even if nonempty

Skorobogatov In cases where \( X(k) = \emptyset \) but \( X(A_k)^{Br} \neq \emptyset \)

\( X(k) \subseteq X(A_k)^{et Br} \subseteq X(A_k)^{Br} \)

Poonen ’08 found \( X \) s.t. \( X(k) = \emptyset \)

but \( X(A_k)^{et Br} \neq \emptyset \)
Poonen's Counterexample

\[ X \xrightarrow{f} C \text{ proper flat} \]

\[ C \text{ curve s.t. } |C(k)| < \infty \]

\[ \dim X = 3 \]

\[ \forall p \in C(k), f^{-1}(p) \text{ Châtelet surface} \]

\[ \text{s.t. } V/\![\![A_{k}]\!\!]^{Br} = \emptyset \]

Easy to show \( X(k) = \emptyset \)

But \( X(\overline{A_{k}})^{\text{et}} \neq \emptyset \)

Key: \( f \) not smooth, i.e. not a fibration
Philosophy of Harpaz-Schlank

\( Br_1 \rightarrow \text{étale descent are étale homotopical} \)

\[ Br(x)[n] \times H^2_{\text{ét}}(x; \mathbb{Z}/n(1)) \]

\( \ker = \text{Pic}/n \)

\( \text{locally const. coeff.} \)

\( \text{finite} \rightarrow \text{étale covers} \rightarrow \Pi_1 \)

HS developed étale htpy obstruction.

**Thm (Harpaz-Schlank)**

étale homotopy obstruction same as étale \( \text{Br} \) for \( k \) n.f.
Thm (C. Schlicht) If \( f: X \to C \)
proper and smooth, \( \ell \underline{\text{tr}}(C) < \infty \),
\( k \) tot. imaginary, then

\[
X(A_k) \overset{\text{EtBr}}{\to} \emptyset
\]

---
idea of proofstrong \( \text{prop} \Rightarrow \text{fibration} \)
can relate \( \text{etBr} \) on \( X \) to
\( \text{etBr} \) on base and on fibers over \( C(k) \)
tool long exact sequence of htpy groups
and equivalence btw \( \text{etBr} \) and htpy
obstructions