Topology and Diophantine Equations

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§1 History

Let k n.f. X/k variety

Q How to determine if $X(k)=\emptyset$?

$$X(k)=\emptyset \Longleftarrow X(\mathbb{A}_k)=\emptyset$$

$$\overset{?}{\Longrightarrow}$$

local-global principle

Lind-Reichardt (40's)

(Selmer) counter-examples. used QR to prove $X(k)=\emptyset$

QR $\Rightarrow$ CFT

Manin ('71) used CFT via Brauer group $H^2_{et}(X; \mathbb{G}_m)$

defined $X(k) \subseteq X(\mathbb{A}_k)^{Br} \subseteq X(\mathbb{A}_k)$

can be empty even if nonempty

Skorobogatov In cases where $X(k) = \emptyset$ but $X(\mathbb{A}_k)^{Br} \neq \emptyset$

$X(k) \subseteq X(\mathbb{A}_k)^{\acute{e}tBr} \subseteq X(\mathbb{A}_k)^{Br}$

Poonen '08 found X s.t. $X(k) = \emptyset$ but $X(\mathbb{A}_k)^{\acute{e}tBr} \neq \emptyset$

§2 Poonen's Counterexample

$$X \xrightarrow{f} C \quad \text{proper flat}$$

C curve s.t. $|C(k)| < \infty$

$\dim X = 3$

$\forall\, p \in C(k)$, $f^{-1}(p)$ Châtelet surface V

s.t. $V(\mathbb{A}_k)^{Br} = \emptyset$

easy to show $X(k) = \emptyset$

<u>but</u> $X(\mathbb{A}_k)^{\text{ét}, Br} \neq \emptyset$

<u>Key</u> f not smooth, i.e. not a fibration

Philosophy of Harpaz-Schlank

Br_1, ét Br_1, finite descent are
étale homotopical

$$Br(X)[n] \neq H^2_{ét}(X; \mathbb{Z}/n(1))$$

$\twoheadleftarrow$ ker $= Pic/n$

↑ locally const. coeff.

ét $\Rightarrow$ finite étale covers $\Rightarrow$ π_1

HS developed étale htpy obstruction.

__Thm__ (Harpaz-Schlank)
étale homotopy obstruction same as étBr
for k n.f.

Thm (C-Schlank) If $f: X \to C$ proper **and** smooth, $|Ш(C)| < \infty$, K tot. imaginary, then

as in Poonen's counterexample

$$X(\mathbb{A}_K)^{\acute{e}tBr} = \emptyset$$

idea of proof sm, prop $\Rightarrow$ fibration

can relate étBr on X to étBr on base and on fibers over $C(k)$

tool long exact sequence of htpy grps and equivalence btw étBr and htpy obstructions