CM Points on $X_0(N)$: Volcanoes and Reality, Extended Edition

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This is the Extended Edition

When preparing for my talk, I had to cut a certain amount of material. Those are the breaks, and I'm sure my talk went better for fitting in the allotted time. On the other hand, some of what got cut is closely related to material that other speakers have discussed. So taking advantage of the online format, I am providing this version of the slides with some of this relevant material put back in IN PURPLE.

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Here I want to discuss work which should lead to a complete solution in the **CM case**.

Motivating Problem: Fibers Over J

Let $X_{/\mathbb{Q}}$ be a modular curve: say $X_0(N)$, $X_1(N)$, $X(N)$ or $X(M,N)$. Let $\pi : X \to X(1)$ be the map to the j-line. Given a closed point $J \in X(1)$, understand the fiber of π over X.

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All hail the triumvirate: torsion subgroups $\iff$ points on modular curves $\iff$ Galois representations.

The CM Case

We will work in the **CM case**. Here much more is known. After pioneering work of Silverberg (1988, 1992) and recent work of Lozano-Robledo, Bourdon, Clark, Pollack, Stankewicz, we are getting close to **complete answers**.

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2019 work of Lozano-Robledo gives lots of information on the mod N and ℓ-adic Galois reps on a CM elliptic curve over $\mathbb{Q}(J)$. His work in progress should do even more.

Imaginary Quadratic Orders

CM setup: an elliptic curve $E_{/\mathbb{C}} \cong \mathbb{C}/\Lambda$ has **complex multiplication** if $\operatorname{End} E = \{\alpha \in \mathbb{C} \mid \alpha\Lambda \subset \Lambda\} \supsetneq \mathbb{Z}$, in which case $\operatorname{End} E$ is an order $\mathcal{O}$ in an imaginary quadratic field K.

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$K = \mathbb{Q}(\Delta_K)$ an imaginary quadratic field. For $\mathfrak{f} \in \mathbb{Z}^+$, unique order $\mathcal{O} = \mathcal{O}(\Delta)$ in K with $[\mathbb{Z}_K : \mathcal{O}] = \mathfrak{f}$, of discriminant $\Delta = \mathfrak{f}^2\Delta_K$. $E_{/\mathbb{C}}$ has **Δ-CM** if $\operatorname{End} E \cong \mathcal{O}(\Delta)$.

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In other words, the j-invariants of Δ-CM elliptic curves form a complete, single Galois orbit. Our favorite j-invariant in this orbit is $j_\Delta := j(\mathbb{C}/\mathcal{O}) \in \mathbb{R}$. More on this later.

Extended Slide: The Main Theorem of CM

In the case $X = X(N)$, which is a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$-Galois covering of $X(1)$ (and cofinal in all modular curves), work of Stevenhagen (and later, Bourdon-Clark and Lozano-Robledo) determines the splitting field of the fiber over $J_\Delta \in X(1)_{/K}$ as an explicit class field. In principle this reduces all the fiber computations on $X \to X(1)_{/K}$ to class field theory.

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Also: want to compute fibers on $X(1)_{/\mathbb{Q}}$. Nailing down difference between $/K$ and $/\mathbb{Q}$ is the hardest part.

Bourdon-Clark

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Why this matters: If F is a number field such that there is a Δ-CM $E_{/F}$ and $\mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \hookrightarrow E(F)$, then there is a closed Δ-CM point $P \in X(M,N)$ and a field embedding $\mathbb{Q}(P) \hookrightarrow F$.

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Transitioning to $X_0(N)$

To classify CM torsion subgroups in degree d, after Bourdon-Clark we still to determine all primitive degrees of closed CM points on $X_1(N)$ and $X(2, 2N)$. (Today: $X_1(N)$.)

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Knowing degrees of all Δ-CM closed points on $X_0(N) \iff$ knowing all degrees of Δ-CM closed points on $X_1(N)$.

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MAIN RESULT: for all Δ (with $\Delta_K < -4$) and all $N \in \mathbb{Z}^+$, we determine the fiber of the $\mathbb{Q}$-morphism $X_0(N) \to X(1)$ over J_Δ and identify the fields $\mathbb{Q}(P)$.

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$$\mathbb{Q}(\mathfrak{f}) := \mathbb{Q}(j_\Delta) = \mathbb{Q}(j(\mathbb{C}/\mathcal{O}(\Delta)))$$

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We have $\mathbb{Q}(\mathfrak{f}_1)\mathbb{Q}(\mathfrak{f}_2) = \mathbb{Q}(\mathrm{lcm}(\mathfrak{f}_1, \mathfrak{f}_2))$.

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$\mathbb{Q}(\mathfrak{f})/\mathbb{Q}$ is Galois iff $h(\Delta) = h_2(\Delta)$, and if j is a Δ-CM j-invariant, then $j \in \mathbb{Q}(\mathfrak{f})$ iff $j \in \mathbb{R}$, since $\mathbb{Q}(\mathfrak{f}) = K(\mathfrak{f})^c$.

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A useful upper bound on $\mathbb{Q}(\varphi)$:

Theorem (Parish, 1989)

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This explains the tensor products of such fields in the next slide.

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(NOT true for $X_1(N)$....isogenies are better.)

Using this and the fact that if $\gcd(\mathfrak{f}_1, \mathfrak{f}_2) = \mathfrak{f}$, then

$$\mathbb{Q}(\mathfrak{f}_1) \otimes_{\mathbb{Q}(\mathfrak{f})} \mathbb{Q}(\mathfrak{f}_2) \cong \mathbb{Q}(\operatorname{lcm}(\mathfrak{f}_1, \mathfrak{f}_2)),$$

$$\mathbb{Q}(\mathfrak{f}_1) \otimes_{\mathbb{Q}(\mathfrak{f})} K(\mathfrak{f}_2) \cong K(\operatorname{lcm}(\mathfrak{f}_1, \mathfrak{f}_2)),$$

$$K(\mathfrak{f}_1) \otimes_{\mathbb{Q}(\mathfrak{f})} K(\mathfrak{f}_2) \cong K(\operatorname{lcm}(\mathfrak{f}_1, \mathfrak{f}_2)) \times K(\operatorname{lcm}(\mathfrak{f}_1, \mathfrak{f}_2)),$$

we reduce to the prime power case.

Um, I was told there would be volcanoes

Fix a prime ℓ. The (K, ℓ)-**isogeny volcano** is a directed multigraph with vertices the j-invariants of K-CM elliptic curves $E_{/\mathbb{C}}$ and with edges $E \to E'$ corresponding to ℓ-isogenies $\varphi : E \to E'$ up to isomorphism on E'. Every edge has an inverse edge, the dual isogeny. The **level** of a vertex is $\mathrm{ord}_\ell(\mathfrak{f})$.

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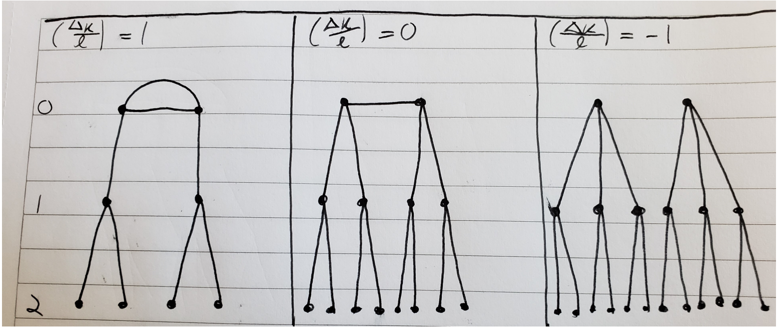
Since ℓ-power isogenies can only change the ℓ-part of $\mathfrak{f}$, the graph breaks up into pieces parameterized by $\mathfrak{f}_0$, the prime-to-ℓ part of $\mathfrak{f}$. Let's also fix $\mathfrak{f}_0$.

Isogeny Volcanoes

The graph has a very simple structure:



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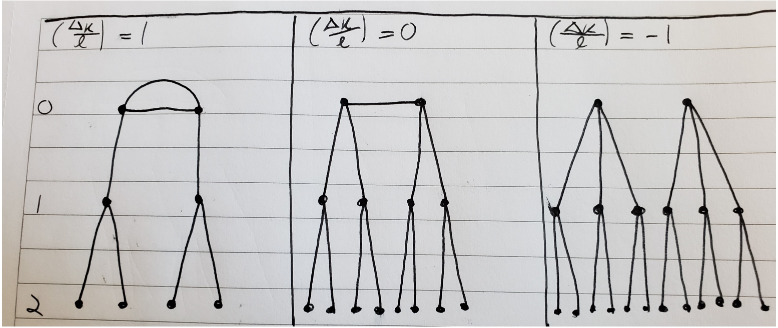
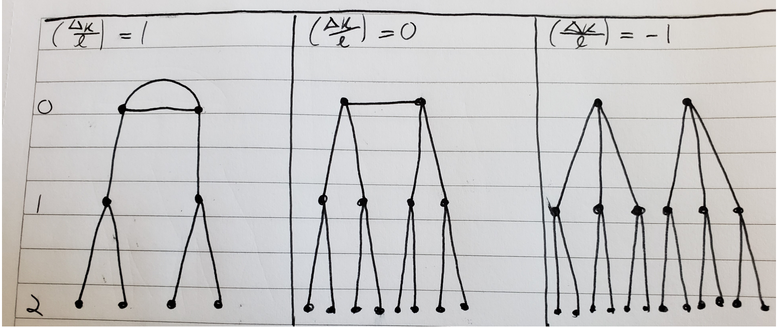


- Every vertex has outward degree $\ell + 1$.
- The set of level 0 vertices is the **surface**. Edges lying within the surface are **horizontal**. Each surface vertex has $1 + \left(\frac{\Delta_K}{\ell}\right)$ horizontal outward edges.

Isogeny Volcanoes II



- The other edges are **ascending**, going from level $L \geq 1$ to level $L - 1$, or **descending**, the inverses of ascending edges.

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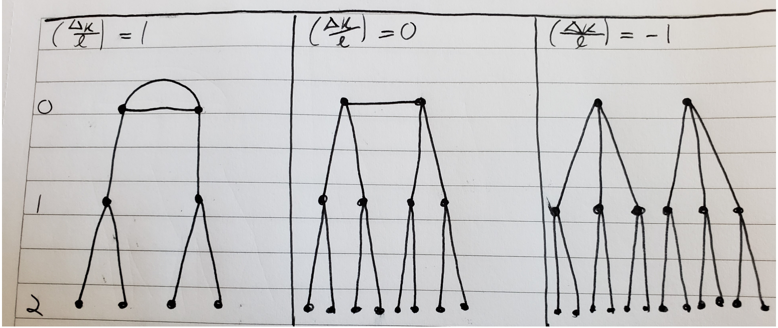
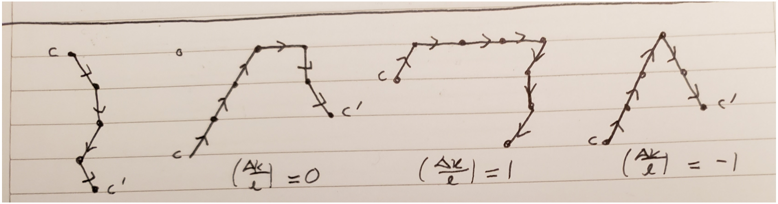


• The other edges are **ascending**, going from level $L \geq 1$ to level $L-1$, or **descending**, the inverses of ascending edges.

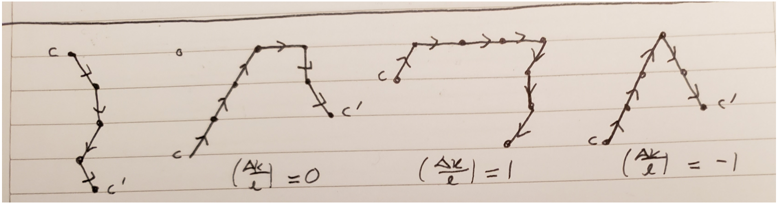
• Every vertex not on the surface has a unique ascending outward edge. From this one deduces the number of descending outward edges every vertex has (it's ℓ away from the surface, and always at least 1).

Paths in Isogeny Volcanoes



Key Fact: Cyclic ℓ^a-isogenies $\varphi : E \rightarrow E' \iff$ length a nonbacktracking paths from $j(E)$ to $j(E')$. Such paths are restricted: they must, ascend, then be horizontal, then descend. (Some parts may have length zero.) So you can count them without real trouble.

Fields of moduli on $X_1(\ell^a)_{/K}$



We can now compute $K(\varphi)$ for any cyclic ℓ^a-isogeny $\varphi : E \to E'$. If E has level c, E' has level c' and $C := \max(c, c')$, then

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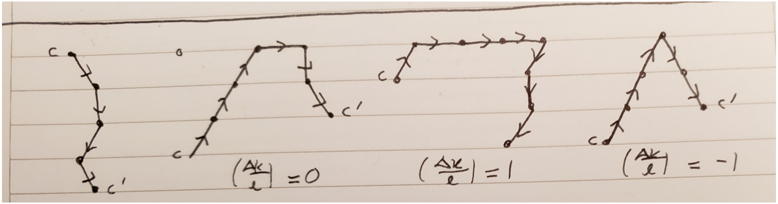

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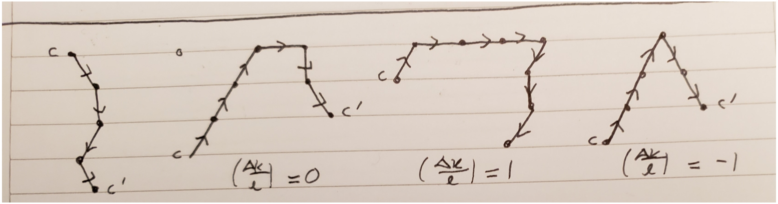

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There is still a nontrivial **counting problem**.

Volcanoes and Reality

We want to work over $\mathbb{Q}$. If $\varphi : E \to E'$ is a cyclic ℓ^a-isogeny with E Δ-CM, WLOG we may assume $j(E) = j_\Delta$. If E has level c and E' has level c', we may assume $c \geq c'$: otherwise switch to $\varphi^\vee$, which has the same field of moduli.

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There is an action of complex conjugation c on the volcano. Call a path **real** if all its edges are c-fixed.

Reality of the Path Determines the Field of Moduli

If the path is not real then either
(i) It contains a non-real surface edge, or
(ii) The terminal vertex is not real.

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Finally, one is left with a refined version of the above combinatorial problem: count real / complex paths, up to closed points. It takes some work...Hope to have a preprint

Thanks!

Thanks for listening, and thanks to the organizers.