Introduction to Sieves Problem Set

Problems with a * are a bit more difficult. If you've taken a course in analytic number theory before, feel free to skip ahead and try some of the * problems!

1 Asymptotic Notation and Arithmetic Functions

Lecture Problems: Each of the following exercises were given in the lecture:

(a) Prove that if $f(x) \sim g(x)$ then $f(x) = O(g(x))$.

(b) Prove that $O(h(x))$ and $o(h(x))$ are additive subgroups in the ring of functions defined for sufficiently large x values satisfying the following properties:

$$f(x) = O(h(x)) \text{ and } g(x) = O(h(x)) \Rightarrow f(x) + g(x) = O(h(x))$$
$$g(x) = O(h(x)) \Rightarrow f(x)g(x) = O(f(x)h(x))$$

and similarly for little-oh.

(c) If $f(x) = O(g(x))$ and $g(x) = O(h(x))$, then $f(x) = O(h(x))$.

(d) Prove that if $f(x) = O(g(x))$ then $\sum_{n \le x} f(n) = O\left(\sum_{n \le x} g(n)\right)$.

(e) Prove that if $f(x) = O(g(x))$ and $y \le x$ is a real number then $\int_y^x f(t)dt = O\left(\int_y^x g(t)dt\right)$.

Problem 1: Prove the following asymptotic relationships as $x \to \infty$:

(a) $\displaystyle\int_2^x \frac{1}{\log t}dt \sim \frac{x}{\log x}$,

(b) $x^n = O(x^m)$ if and only if $n \le m$,

(c) $(\log x)^r = o(x^\epsilon)$ for all real numbers r and all positive real numbers $\epsilon > 0$,

(d) $e^{1/x} = 1 + O\left(x^{-1}\right)$.

Problem 2: For each of the following arithmetic functions (1) graph the function in CoCalc, (2) graph the smoothed out $\sum_{n \le x} f(n)$ version of the function in CoCalc, and (3) make an educated guess for the order of magnitude of the main term, and compare the graphs in CoCalc to see if they appear to be close.

(a) $\nu(n) = \#\{\text{prime divisors of } n\}$

(b) $\sigma_1(n) = \sum_{d|n} d$

(c) $\phi(n) = \#\{1 \le d \le n : \gcd(d, n) = 1\}$

Problem 3: A *palindromic number* is a number that reads the same forwards as backwards (examples: 11, 55, 1001, etc).

(a) Find an explicit formula for how many palindomic numbers there are with exactly n digits.

(b) Determine a differentiable function $f(n)$ which is asymptotic to $\#\{\text{palindromic numbers} \le 10^n\}$ as $n \to \infty$.

(c) What can you conclude about asymptotic rate of growth of $\#\{\text{palindromic numbers} \le x\}$?

2 Abel Summation

Problem 1: Determine a differentiable function which is asymptotic to each of the following functions as $x \to \infty$. Additionally, be explicit about the order of magnitude of the error term.

(a) $\displaystyle\sum_{n \le x} \log n$

(b) $\displaystyle\sum_{n \le x} \frac{d(n)}{n}$

(c) $\displaystyle\sum_{n \le x} n^2$

(d) $\displaystyle\sum_{n > x} \frac{1}{n^2}$

***Problem 2:** The Riemann Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for s a complex number with $\mathrm{Re}(s) > 1$. Reminder: if z is complex, then $|t^z| = |t|^{\mathrm{Re}(z)}$. For a shorter problem, try to solve it only for s a real number instead.

(a) Prove that the following integral is convergent for $\mathrm{Re}(s) > 0$ with

$$\int_x^{\infty} \frac{\lfloor t \rfloor - t}{t^{s+1}} dt = O\left(x^{-s}\right).$$

(b) Using Abel summation and part (a), prove that

$$\sum_{n \le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \left(\frac{s}{s-1} + s \int_1^{\infty} \frac{\lfloor t \rfloor - t}{t^{s+1}} dt \right) + O(x^{-s})$$

for $\mathrm{Re}(s) > 0$ and $s \neq 1$

(c) Conclude that if $\mathrm{Re}(s) > 1$ then

$$\zeta(s) = \frac{s}{s-1} + s \int_1^{\infty} \frac{\lfloor t \rfloor - t}{t^{s+1}} dt = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right)$$

These formulas make sense for $s \neq 1$ and $\mathrm{Re}(s) > 0$, and can be used to "analytically continue" $\zeta(s)$ to this region.

(d) Using a similar process to parts (a), (b), and (c), prove that

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where

$$\gamma = 1 + \int_1^{\infty} \frac{\lfloor t \rfloor - t}{t^2} dt = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n} - \log x \right)$$

is the Euler-Mascheroni constant.

***Problem 3:** A *Dirichlet character* mod N is an arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ which satisfies $\chi(ab) = \chi(a)\chi(b)$, $\chi(1) = 1$, and $\chi(n) = 0$ when $\gcd(n, N) \neq 1$ for which $a \equiv b \mod N$ implies $\chi(a) = \chi(b)$. For a shorter problem, try to solve this problem only for $N = 3$, 4, and 5.

(a) The trivial character mod N is given by

$$\chi_0(n) = \begin{cases} 1 & \gcd(n, N) = 1 \\ 0 & \text{else.} \end{cases}$$

Prove that

$$\sum_{n=1}^{N} \chi_0(n) = \phi(N),$$

and so

$$\sum_{n \leq x} \chi_0(n) = \frac{\phi(N)}{N} x + O(1).$$

(b) Prove that for a nontirival Dirichlet character mod N

$$\sum_{n=1}^{N} \chi(n) = 0\,,$$

and conclude that

$$\sum_{n \leq x} \chi(n) = O(1)\,.$$

(c) The *Dirichlet L-function* associated to a character χ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\,.$$

Prove that this series converges absolutely for $\mathrm{Re}(s) > 1$.

(d) Using Abel summation, prove that for each nontrivial character χ mod N, the L-function $L(s, \chi)$ converges conditionally for $\mathrm{Re}(s) > 0$.

3 Möbius Function

Lecture Problems: Each of the following exercises were given in the lecture:

(a) Prove that if $F(x) = \sum_{n \le x} G(x/n)$ then $G(x) = \sum_{n \le x} \mu(n) F(x/n)$.

Problem 1: The following steps are an alternate formulation for the asymptotic size of the set of squarefree numbers:

(a) Prove that

$$\sum_{d^2 | n} \mu(d) = \begin{cases} 1 & n \text{ is squarefree} \\ 0 & \text{else} \end{cases}$$

(b) Part (a) implies that

$$\#\{\text{squarefree } n \le x\} = \sum_{n \le x} \sum_{d^2 | n} \mu(d).$$

Switch the order of the summations in order to compute the main term and error term.

(c) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)}$.

(d) Generalize this approach to prove that

$$\#\{k\text{-power free } n \le x\} \sim \frac{1}{\zeta(k)} x + O(x^{1/k})$$

for any integer $k > 1$, where we say n is *k-power free* if $d^k \mid n$ implies $d = 1$.

Problem 2: Let $\phi(n) = \#\{1 \le d < n : \gcd(d, n) = 1\}$ be Euler's ϕ-function.

(a) Prove that

$$\phi(n) = \sum_{d | n} \mu(d) \frac{n}{d}.$$

(b) Using part (a), conclude that

$$\sum_{n \le x} \phi(n) = \frac{1}{2\zeta(2)} x^2 + O(x \log x).$$

4 Prime Numbers

Lecture Problems:

(a) Prove that $\pi(x) = O\left(\frac{x}{\log x}\right)$ if and only if $\theta(x) = \sum_{p \le x} \log p = O(x)$.

(b) $\sum_{n \le x} \log(n) = x \log x - x + O(\log x)$ (also an earlier exercise).

(c) Prove that $\sum_{p \le x} \frac{1}{p} = \log \log x + O(1)$ using Abel summation with $f(t) = (\log t)^{-1}$.

Problem 1: Let $\nu(n) = \#\{\text{distinct prime divisors of } n\} = \sum_{p \mid n} 1$. Prove that

$$\sum_{n \le x} \nu(n) = x \log \log x + O(x).$$

5 Sieve of Eratosthenes

Problem 1: We modify the sieve of Eratosthenes slightly to answer some questions about cubefree numbers.

(a) Let $\mathcal{A} = \{n \leq x\}$, $\omega(p) = 2$, and $\mathcal{A}_p = \{n \leq x : n \equiv 0 \text{ or } -2 \mod p^3\}$. Define $\omega(d) = \prod_{p|d} \omega(p)$ and $\mathcal{A}_d = \bigcap_{p|d} \mathcal{A}_p$ and prove that

$$\#\mathcal{A}_d = \frac{\omega(d)}{d^3} x + O(\omega(d)).$$

Moreover, prove that if $d > (x+2)^{2/3}$ then $\#\mathcal{A}_d = 0$.

(b) As usual, define $S(\mathcal{A}, \mathcal{P}, z) = \#\left(\mathcal{A} \setminus \bigcup_{p|P(z)} \mathcal{A}_p\right)$. Prove that

$$\#\{n \leq x : n \text{ and } n+2 \text{ are cubefree}\} = S(\mathcal{A}, \mathcal{P}, (x+2)^{2/3})$$

(c) Following the argument for the sieve of Eratosthenes, prove that

$$S(\mathcal{A}, \mathcal{P}, z) = xW(z) + O\left(x^{2/3} (\log z) \exp\left(-\frac{2 \log x}{3 \log z}\right)\right),$$

where

$$W(z) = \prod_{p<z} \left(1 - \frac{2}{p^3}\right).$$

(d) Prove that as $z \to \infty$

$$W(z) = \prod_{p} \left(1 - \frac{2}{p^3}\right) + o(1),$$

where the infinite product is absoutley convergent.

(e) Using all of the above, conclude that

$$\#\{n \leq x : n \text{ and } n+2 \text{ are cubefree}\} \sim x \prod_{p} \left(1 - \frac{2}{p^3}\right).$$

*(f) Improve the bounds on the error terms in parts (c) and (d), so that you can find a bound for the error term in part (e).

***Problem 2:** Follow a similar process to Problem 1 to compute the asymptotic main term of

$$\#\{n \leq x : n^2 + 1 \text{ is cubefree}\}.$$

The corresponding question for prime numbers is an open problem, we do not know if there are infinitely many primes of the form $n^2 + 1$.

Problem 3: We introduce some of the ideas behind the first Hardy–Littlewood conjecture on prime constellations.

(a) A k-tuple of nonnegative integers $(k_1, k_2, ..., k_r)$ is called *admissible* if $\{k_1, k_2, ..., k_r \mod N\} \neq \{0, 1, 2, ..., N-1\}$ for any $N > 1$. For a non-admissible k-tuple, prove that

$$\#\{n : n + k_1, n + k_2, ..., n + k_r \text{ are all prime}\} \leq 1.$$

(b) For an admissible k-tuple $(k_1, ..., k_r)$, define

$$\omega(p) = \#\{k_1, ..., k_r \mod p\}.$$

apply the sieve of Eratosthenes to show that

$$\#\{n \leq x : n + k_1, n + k_2, ..., n + k_r \text{ are all prime}\} = x \prod_{p<x} \left(1 - \frac{\omega(p)}{p}\right) + O\,(\text{something})$$

(c) Ignoring the error term, prove that there exist constants c_1 and c_2 such that

$$c_1 \frac{x}{(\log x)^r} \leq x \prod_{p<x} \left(1 - \frac{\omega(p)}{p}\right) \leq c_2 \frac{x}{(\log x)^r}.$$

*(d) Prove that there exists a constant c_k such that

$$x \prod_{p<x} \left(1 - \frac{\omega(p)}{p}\right) \sim c_k \frac{x}{(\log x)^r}.$$

The first Hardy–Littlewood conjecture posits that for any admissible k-tuple, $\#\{n \leq x : n + k_1, n + k_2, ..., n + k_r \text{ are all prime}\}$ is asymptotic to this function (with an explicit value fo c_k). This conjecture is still open, why do the steps in this problem FAIL to prove the conjecture?

*(e) Using the ideas in this problem and in Problem 1, can you propose the statement of a "Hardy–Littlewood conjecture for k-power free numbers"? Can the sieve of Eratosthenes be used to prove this conjecture? What about one of Brun's sieves (if you revisit this question later)?

Problem 4: Assume the validaty of the Hardy-Littlewood conjecture for this problem (stated in Problem 2 (d)), and prove the following:

(a) Prove that the Hardy–Littlewood conjecture implies a generalization of a theorem of Brun on twin primes:

$$\sum_{\substack{n \leq x \\ n+k_1, ..., n+k_r \text{ prime}}} \frac{1}{n} < \infty.$$

(b) Prove that the Hardy–Littlewood conjecture implies that

$$\sum_{\substack{n \leq x \\ n+k_1, ..., n+k_r \text{ prime}}} \frac{\log n}{n}$$

diverges.

(c) Find a function $f(x)$ for which $\lim_{x\to\infty} f(x) = \infty$ such that any set $\mathcal{B}$ of integers satisfying $\#\{n \in \mathcal{B} : n \leq x\} \sim f(x)$ also satisfies

$$\sum_{n \in \mathcal{B}} \frac{\log n}{n} < \infty.$$

6 Brun's Sieves

Lecture Problems:

(a) Prove that $\sum_{r=0}^{k} \binom{n}{r}(-1)^r = \binom{n-1}{k}(-1)^k$.

Problem 1: For each positive integer k, find an upper bound for the number of primes p for which $p+k$ is also a prime which is strong enough to conclude that

$$\sum_{\substack{p \\ p+k \text{ prime}}} \frac{1}{p} < \infty.$$

Can you compute the value of this sum for any choices of k?

Problem 2: Brun used his powerful sieving techniques to prove that when $\mathcal{A}_p = \{n \leq x : n \equiv 0 \text{ or } -2 \mod p\}$, sieving produces the bounds

$$c_1 x W(z) + O(z^\theta) \leq S(\mathcal{A}, \mathcal{P}, z) \leq c_2 x W(z) + O(z^{\theta+1})$$

for explicit constants c_1 and c_2, and some $\theta < 8$.

(a) Using this result, prove that there exist positive constants C_1, C_2, and ϵ such that

$$C_1 \frac{x}{(\log x)^2} + O(x^{1-\epsilon}) \leq \#\{n \leq x : n \text{ and } n+2 \text{ have at most 8 prime factors}\} \leq C_2 \frac{x}{(\log x)^2} + O(x^{1-\epsilon}).$$

(b) Following your argument in part (a), how small would θ need to be in order to prove the twin prime conjecture?