p-adic functions on $\mathbb{Z}_p$. Lecture 4.

Today: "p-adic version" of Riemann zeta function
called Kubota-Leopoldt p-adic L-function $L_p(s)$

* $L_p(s)$ "p-adically interpolates" special values $\zeta(-n) \in \mathbb{Q}$ for all odd n.
* meaning: in a systematic way to explain

Kummer's congruence: Let $p \geq 3$ be a prime. For any $k \geq 1$,
if $n_1 \equiv n_2 \mod (p-1)p^{k-1}$ and $p-1 \nmid n_i+1$ and $n_1, n_2 \geq k$, then
$$\zeta(-n_1) \equiv \zeta(-n_2) \mod p^k.$$

This was proved in 1800s, but only got understood more conceptually in 1960s during the development of p-adic theory, p-adic L-functions, ...

Example: $p=7$, $k=2$ $3 \equiv 45 \mod 42$

$$\zeta(-3) = \frac{1}{120} = 1 + 4\cdot 7 + 6\cdot 7^2 + \cdots$$ congruent mod. 7^2

$$\zeta(-45) = \frac{-2593265702582226796860 7}{564} = 1 + 4\cdot 7 + 3\cdot 7^3 + \cdots$$

Non-example: $p=7$, $k=2$, $5 \equiv 47 \mod 42$ BUT $6 \mid \underline{5+1}$

$$\zeta(-5) = -\frac{1}{252} = \frac{6}{7} + 4 + 3\cdot 7 + \cdots \longleftarrow$$ note the "denominator"

$$\zeta(-47) = \frac{5609403368997817686249127547}{2227680} = \frac{6}{7} + 5 + 2\cdot 7 + \cdots$$

Step 1: Give an algebraic way to access the special values $\zeta(-n)$ (Bernoulli numbers)

* Recall: $\Gamma(s) := \int_0^{+\infty} e^{-t} t^s \frac{dt}{t}$

Then $$\Gamma(s)\zeta(s) = \int_0^{+\infty} e^{-t} t^s \cdot \sum_{n\geq 1} \frac{1}{n^s} \cdot \frac{dt}{t}$$ converges absolutely when $\mathrm{Re}\, s \gg 0$

$$= \sum \int^{+\infty} e^{-t} \left(t\right)^s \frac{dt}{}$$

$$= \sum_{n\geq 1}\int_0^{+\infty} e^{-t}\cdot\left(\frac{t}{n}\right)^s \frac{dt}{t}$$

$$\overset{t\mapsto nt}{=} \sum_{n\geq 1}\int_0^{+\infty} e^{-nt}\cdot t^s\,\frac{dt}{t}$$

$$= \int_0^{+\infty}\frac{e^{-t}}{1-e^{-t}}\,t^s\,\frac{dt}{t} = \int_0^{+\infty}\frac{1}{e^t-1}\,t^s\,\frac{dt}{t}$$

Or $\displaystyle \zeta(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{1}{e^t-1}\,t^s\,\frac{dt}{t}$

<u>Technical Lemma</u> If $f:\mathbb{R}_{>0}\to\mathbb{C}$ is a smooth function on $\mathbb{R}_{>0}$, rapidly decreasing (i.e. $t^n f(t)\to 0$ when $t\to+\infty$, for all $n\in\mathbb{N}$) and $t f(t)$ bounded as $t\to 0$

then $\displaystyle L(f,s) := \frac{1}{\Gamma(s)}\int_0^{+\infty} f(t)\,t^s\,\frac{dt}{t}$ $\quad \mathrm{Re}(s)>0$

as above, think about $f(t)=\dfrac{1}{e^t-1}$

has an analytic continuation to $\mathbb{C}$, and

$$L(f,-n) = (-1)^n f^{(n)}(0) \qquad (\star)$$

Proof of $(\star)$ minus convergence issue (assume that $f(t)=0$ when $t\gg 0$)

$$L(f,s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} f(t)\,t^s\,\frac{dt}{t}$$

$$= \frac{1}{\Gamma(s)}\underbrace{\left(f(t)\frac{t^s}{s}\right)\Big|_0^{+\infty}}_{=0} - \frac{1}{\underbrace{s\Gamma(s)}_{=\Gamma(s+1)}}\int_0^{+\infty} f'(t)\,t^{s+1}\,\frac{dt}{t}$$

$$= -L(f', s+1) = \cdots = (-1)^{n+1} L(f^{(n)}, s+n+1)$$

$$\Rightarrow L(f,-n) = (-1)^{n+1} L(f^{(n+1)}, 1) = (-1)^{n+1}\int_0^{+\infty} f^{(n+1)}(t)\,dt = (-1)^n f^{(n)}(0).$$

Now, apply the lemma to $\displaystyle f(t) = \frac{1}{e^t-1} = \frac{1}{t} + \sum_{n\geq 0} B'_n \frac{t^n}{n!}$

(compare to classical theory:

$$\frac{t}{e^t-1} = 1+\sum_{n\geq 0} B_n'\cdot(n+1)\cdot\frac{t^{n+1}}{(n+1)!}$$

Then $(n+1)\cdot B_n'$ is the usual Bernoulli number.)

<u>Note</u>: $\frac{1}{e^t-1}+\frac{1}{e^{-t}-1} = \frac{1}{e^t-1}+\frac{e^t}{1-e^t} = \frac{1-e^t}{e^t-1} = -1$. so $B_{\text{even}}'=0$

$$\Rightarrow \zeta(-n) = L(f,-n) = (-1)^n\cdot f^{(n)}(0) = (-1)^n B_n'$$

Take n odd, $n\geq 3 \Rightarrow \zeta(-n) = -B_n'$

<u>Step 2</u>: p-adic interpolation

<u>Technical note</u>: $\zeta(s)$ has a pole at $s=1$

So we expect its p-adic analogue has similar property

$\rightsquigarrow$ will fix $a\in\mathbb{Z}$ not divisible by p and interpolate

$(1-a^{1+n})\zeta(-n)$ instead

$\uparrow$ is zero when $n=-1$.

<u>Theorem</u>. Consider a measure μ_a on $\mathbb{Z}_p$ s.t.

$$A_{\mu_a}(T) = \frac{1}{T} - \frac{a}{(1+T)^a-1} \in \mathbb{Z}_p[\![T]\!]$$

Then $$\int_{\mathbb{Z}_p} x^n\, d\mu_a(x) = (a^{1+n}-1)\zeta(-n)$$ for all n odd.

Proof:
$$\int_{\mathbb{Z}_p} x^n\, d\mu_a(x) = \left(\frac{d}{dt}\right)^n\Big|_{t=0}\left(\int_{\mathbb{Z}_p} e^{tx}\, d\mu_a(x)\right)$$
$$= \left(\frac{d}{dt}\right)^n\Big|_{t=0}\left(\int_{\mathbb{Z}_p} (1+e^t-1)^x\, d\mu_a(x)\right)$$
$$= \left(\frac{d}{dt}\right)^n\Big|_{t=0} A_{\mu_a}(e^t-1) = f_a^{(n)}(0)$$

for $f_a(t) = \dfrac{1}{e^t-1} - \dfrac{a}{e^{at}-1}$

<u>Note</u>: $f_a^{(n)}(0) = \underbrace{\left(\dfrac{1}{e^t-1}\right)^{(n)}\Bigg|_{t=0}}_{\substack{\parallel \\ -\zeta(-n)}} - \underbrace{\left(\dfrac{a}{e^{at}-1}\right)^{(n)}\Bigg|_{t=0}}_{\substack{\parallel \\ -a^{n+1}\zeta(-n)}} = (a^{n+1}-1)\zeta(-n)$

· Note $L\left(\dfrac{a}{e^{at}-1}, s\right) = \dfrac{1}{\Gamma(s)}\displaystyle\int_0^{+\infty} \dfrac{a}{e^{at}-1} t^s \dfrac{dt}{t}$

$$= \frac{1}{a^{s-1}} \cdot \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1}{e^{at}-1} (at)^s \frac{dt}{t}$$

$\Rightarrow L\left(\frac{a}{e^{at}-1}, -n\right) = -a^{1+n}\zeta(-n)$ $\square$

Proof of Kummer's congruence: For any $k \geq 1$,
if $n_1 \equiv n_2 \mod (p-1)p^{k-1}$ and $p-1 \nmid n_i+1$ and $n_1, n_2 \geq k$, then
$\zeta(-n_1) \equiv \zeta(-n_2) \mod p^k$.

By theorem above, we have

$$(a^{1+n_1}-1)\zeta(-n_1) = \underbrace{\int_{\mathbb{Z}_p} x^{n_1} d\mu_a \quad \text{vs.} \quad \int_{\mathbb{Z}_p} x^{n_2} d\mu_a}_{} = (a^{1+n_2}-1)\zeta(-n_2)$$

<u>Note</u>: for every $x \in \mathbb{Z}_p$, $x^{n_1} \equiv x^{n_2} \mod p^k$

Thus, $v_p\left(\displaystyle\int_{\mathbb{Z}_p} (x^{n_1} - x^{n_2}) d\mu_a\right) \geq k$.

Also if we choose a to be a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ when modulo p,

$$\frac{1-a^{1+n_1}}{1-a^{1+n_2}} \equiv 1 \mod p^k$$

$\Rightarrow \zeta(-n_1) \equiv \zeta(-n_2) \mod p^k$.

Step 3: Restriction to $\mathbb{Z}_p^\times$.

Definition: The p-adic zeta function (Kubota-Leopoldt p-adic L-function) is the measure μ_a restricted to $C(\mathbb{Z}_p^\times, \mathbb{Q}_p)$, denoted by $\mu_{KL,a}$

Why restricted to $\mathbb{Z}_p^\times$?

- In the above proof, over $p\mathbb{Z}_p$, we need $x^{n_1} \equiv x^{n_2} \bmod p^k$ but this is achieved by asking $n_i \geqslant p$. (not natural).
- More philosophical reason: $\mathrm{Gal}_{\mathbb{Q}_p}^{\text{pro-}p\text{-ab}} = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$
  A p-adic L-function is canonically a measure on this

Theorem: For every $n \geqslant 2$,

$$\int_{\mathbb{Z}_p^\times} x^n \mu_{KL,a} = (-1)^n (1-a^{n+1}) \underline{\underline{(1-p^n)\zeta(-n)}}$$

$\|$ analytic continuation of $\zeta^{(p)}(s) = \prod_{\ell \neq p} \frac{1}{1-\ell^{-s}}$ to $s=-n$.

(namely with the factor at p removed).

If one really prefers a "function on $\mathbb{Z}_p$", one can fix $i \in (\mathbb{Z}/p\mathbb{Z})^\times$ define

$$\begin{aligned} \mathbb{Z}_p &\longrightarrow \mathbb{Q}_p \\ s &\longmapsto \int_{\mathbb{Z}_p^\times} [\bar{x}]^i (x^{p-1})^{\frac{s}{p-1}} \mu_{KL,a} \end{aligned}$$

$\longleftrightarrow$ zeta values at $-n$ with some modification

This is a continuous function on $s \in \mathbb{Z}_p$.

But this is not the correct formulation.

Slogan: The correct p-adic version of L-function interpolates values of the usual L-function with the factor at p slightly changed

& is often a measure on $\mathbb{Z}_p^\times$, & satisfies

$$\underbrace{\int_{\mathbb{Z}_p^\times} x^n}\mu_{\mathcal{L}} = (\text{simple factors}) \cdot \zeta^{(p)}(-n)$$

corresponds to the cyclotomic character

shifting the $\mathcal{L}$-function by n.

can also put in other char

$\psi : \mathbb{Z}_p^\times \to \mathbb{C}^\times \longleftrightarrow$ Dirichlet L-function

$$\int_{\mathbb{Z}_p^\times} \psi(x) x^n \mu_{\mathcal{L}} = (\text{simple factor}) \cdot L^{(p)}(\psi^{-1}, s=-n)$$