p-adic functions on $\mathbb{Z}_p$. Lecture 3.

<u>Outline</u>: p-adic Banach space

Mahler basis for $C(\mathbb{Z}_p; \mathbb{Q}_p)$

p-adic measures on $\mathbb{Z}_p$

<u>Slogan</u>: Many theories over $\mathbb{R}$ have analogues over $\mathbb{Q}_p$

<u>Recall</u>: A Banach space V (over $\mathbb{R}$) is an $\mathbb{R}$-vector space,

complete with respect to a norm $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$

s.t. (1) $\|av\| = |a| \cdot \|v\|$ for $a \in \mathbb{R}$, $v \in V$

(2) $\|v+w\| \leq \|v\| + \|w\|$

(3) $\|v\| = 0 \iff v = 0$.

<u>Example</u>: $C([0,1], \mathbb{R}) = \{\text{continuous functions } f : [0,1] \to \mathbb{R}\}$

$$\|f\| = \max_{x \in [0,1]} |f(x)|.$$

<u>Def'n</u>: A <u>(p-adic) Banach space</u> V over $\mathbb{Q}_p$ is a $\mathbb{Q}_p$-vector space,

complete with respect to a norm $\|\cdot\| : V \longrightarrow \mathbb{R}_{\geq 0}$

s.t. (1) $\|av\| = |a|_p \cdot \|v\|$ for $a \in \mathbb{Q}_p$, $v \in V$

(2) $\|v+w\| \leq \max\{\|v\|, \|w\|\}$ $\forall v, w \in V$

(3) $\|v\| = 0 \iff v = 0$.

We say that a Banach space V has an <u>orthonormal basis</u> $\{e_i\}_{i \in I}$ with $e_i \in V$

if (1) $\|e_i\| = 1$ for each $i \in I$

(2) every vector $v \in V$ can be written <u>uniquely</u> as converging limit

$$v = \sum_{i \in I} a_i e_i \quad \text{for } a_i \in \mathbb{Q}_p.$$

(This requires (if $I=\mathbb{N}$), $|a_i|_p \to 0$ as $i \to \infty$
In general, this means that for any ε, there are only finitely many $i \in I$ s.t. $|a_i|_p < \varepsilon$.)

(3) $$\left\| \sum_{i \in I} a_i e_i \right\| = \max_{i \in I} \{ |a_i| \}$$

Rmk: This concept is the p-adic analogue of Hilbert space over $\mathbb{R}$, except that the inner product norm $\|v\|^2 + \|w\|^2 = \|v+w\|^2$ for orthogonal vectors is replaced by the condition $\|v+w\| = \max\{\|v\|, \|w\|\}$ for "orthogonal vectors"
more like L^1-space

Typical example: $\ell^1 = \{$ sequence of numbers $a_1, a_2, \cdots$ in $\mathbb{Q}_p$ s.t. $\lim_{n\to\infty} |a_n|_p = 0\}$

For a sequence $(a_n)_{n\in\mathbb{N}}$, define $\|(a_n)_{n\in\mathbb{N}}\| := \max_n \{|a_n|_p\}$

Orthonormal basis: $e_n = (0, 0, \cdots, \underset{\substack{\downarrow \\ n^{th}\text{ place}}}{1}, 0, \cdots)$

Then $(a_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} a_n \cdot e_n$ with norm $\max_n \{|a_n|_p\}$.

Theorem. $C(\mathbb{Z}_p; \mathbb{Q}_p) := \{$ continuous functions $f: \mathbb{Z}_p \to \mathbb{Q}_p \}$

$$\|f\| := \max_{a \in \mathbb{Z}_p} \{ |f(a)|_p \}$$
← This norm is well-def'd b/c $\mathbb{Z}_p$ is compact, so $f(\mathbb{Z}_p)$ is bounded

is a Banach space, with orthonormal basis.

Quick check properties for Banach spaces:

(0) complete for the norm

(1) $\|af\| = |a|_p \cdot \|f\|$

(2) $\|f+g\| \le \max\{\|f\|, \|g\|\}$

(3) $\|f\| = 0 \Leftrightarrow f = 0$.

Question: Examples of continuous functions on $\mathbb{Z}_p$ (with values in $\mathbb{Z}_p$)?

- Polynomials with integer coeffs.
- Some power series , e.g. $\exp(px)$) limit of the previous one.

$1, x, x^2, \cdots, x^p, x^{p+1}$

E.g. $\binom{x}{p} = \dfrac{x(x-1)\cdots(x-(p-1))}{p!}$. This functions take value in $\mathbb{Z}_p$ despite the denominator p.

Proof: Method 1: Every $x \in \mathbb{Z}_p$ belongs to one of $p\mathbb{Z}_p, 1+p\mathbb{Z}_p, \cdots, (p-1)+p\mathbb{Z}_p$ say $x \in a + p\mathbb{Z}_p$, then p divides the factor $x-a$

$\Rightarrow \binom{x}{p} \in \mathbb{Z}_p$ ($(p-1)!$ is invertible in $\mathbb{Z}_p$.)

Method 2: It is clear that if $x \in \mathbb{N}$, then $\binom{x}{p} \in \mathbb{N}$ (by defn of binomial coeffs)

It's also clear that $\binom{x}{p}$ is continuous.

But $\mathbb{N}$ is dense in $\mathbb{Z}_p \Rightarrow$ all of $\binom{x}{p}$ takes values in $\mathbb{Z}_p$. $\square$

Restate the theorem $\left\{1, x, \binom{x}{2}, \cdots, \binom{x}{n}, \cdots\right\}$ is an orthonormal basis of $C(\mathbb{Z}_p; \mathbb{Q}_p)$.

called Mahler basis

In particular, every continuous function $f: \mathbb{Z}_p \to \mathbb{Q}_p$ can be written uniquely as

$$(*) \quad f(x) = \sum_{n \geq 0} a_n \binom{x}{n} \quad \text{with } |a_n|_p \to 0.$$

called the Mahler expansion

Proof: Step 1: Get the expression $(*)$. Assume that it holds

- Compare at $x=0 \Rightarrow a_0 = f(0)$

—— $x = 1 \Rightarrow a_0 + a_1 = f(1) \Rightarrow a_1 = f(1) - f(0)$

—— $x = 2 \Rightarrow a_0 + 2a_1 + a_2 = f(2) \Rightarrow a_2 = f(2) - 2f(1) + f(0)$

Explain how this looks very much like the Taylor expansion.

Define $f^{[1]}(x) := f(x+1) - f(x)$

$f^{[n+1]}(x) := f^{[n]}(x+1) - f^{[n]}(x)$ some literatures write $\Delta^{(n+1)}(f)(x)$

Assume $(*)$ holds, $f^{[1]}(x) = \sum_{n\geq 0} a_n\left(\binom{x+1}{n} - \binom{x}{n}\right) = \sum_{n\geq 0} a_n \binom{x}{n-1}$

& thus by induction, $f^{[m]}(x) = \sum_{n\geq 0} a_n \cdot \binom{x}{n-m}$

So evaluating at $x=0 \Rightarrow a_m = f^{[m]}(0)$

(We've proved that the expression $(*)$ is unique, if it exists.)

Step 2 For $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, as $m \to \infty$, $|f^{[m]}(0)| \to 0$.

Rough idea: will show that for $k \gg 0$, $\|f^{[p^k]}\| \leq |p| \cdot \|f\|$

b/c $\mathbb{Z}_p$ is compact so f is uniformly continuous

$$f^{[p^k]}(x) = \underbrace{f(x+p^k) - f(x)}_{\text{more div by } p \text{ if } k \gg 0.} + \underbrace{\sum_{i=1}^{p^k-1} (-1)^i \binom{p^k}{i} f(x+p^k-i)}_{\text{all more div. by } p}$$

Step3: For $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, setting $a_m := f^{[m]}(0)$ $\forall m \geq 0$

then $f(x) = \sum_{n\geq 0} a_n \binom{x}{n}$

Indeed, $\sum_{n\geq 0} a_n \binom{x}{n}$ converges in $\mathbb{Z}_p$

& $f(x) - \sum_{n\geq 0} a_n \binom{x}{n}$ is zero at all $x \in \mathbb{Z}_{\geq 0}$ ← dense in $\mathbb{Z}_p$

$\Rightarrow f(x) = \sum_{n\geq 0} a_n \binom{x}{n}$.

Step 4: $\left\|\binom{x}{n}\right\| = 1$ & $\|f\| = \max_n |a_n|_p$

* Have shown that for $x \in \mathbb{Z}_p$, $\binom{x}{n} \in \mathbb{Z}_p \Rightarrow \left\|\binom{x}{n}\right\| \leq 1$.

  & when $x = n$, $\binom{x}{n} = 1 \Rightarrow \left\|\binom{x}{n}\right\| = 1$.

* By expression $(*)$ $\|f\| \leq \max_n |a_n|_p \cdot \left\|\binom{x}{n}\right\| = \max_n |a_n|_p$

  On the other hand, each a_n is an integer linear combination of values of f.

  $\Rightarrow |a_n|_p \leq \|f\|$ for every n.

  $\Rightarrow \|f\| = \max_n |a_n|_p$.

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p-adic measure:

A measure on $\mathbb{Z}_p$ with values in $\mathbb{Q}_p$ is a continuous linear map

$$C(\mathbb{Z}_p;\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$$
$$f(x) \longmapsto \int_{\mathbb{Z}_p} f(x)\, d\mu.$$

$$\mathcal{D}(\mathbb{Z}_p;\mathbb{Q}_p) := \{\text{all measures on } \mathbb{Z}_p \text{ with values in } \mathbb{Q}_p\}$$

looks like an $\ell^\infty(\mathbb{N})$.

Theorem. All measures on $\mathbb{Z}_p$ are given by a power series $A_\mu(T) \in \mathbb{Z}_p[\![T]\!][\frac{1}{p}]$

$$A_\mu(T) = b_0 + b_1 T + b_2 T^2 + \cdots \quad \text{with } b_i \in \mathbb{Q}_p \text{, bounded.}$$

Explicitly, for $f(x) = \sum_{n\geq 0} a_n \binom{x}{n} \in C(\mathbb{Z}_p;\mathbb{Q}_p)$

$$\int_{\mathbb{Z}_p} f(x)\, \mu = a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots$$

Amice transform: Given a measure μ, the associate power series $A_\mu(T)$ is

$$A_\mu(T) = \int (1+T)^x\, d\mu$$

$$(1+T)^m = \sum_{n\geq 0} \binom{m}{n} T^n$$

$$= \int \sum_{n\geq 0} \binom{x}{n} T^n\, d\mu$$

$$= \sum_{n\geq 0} T^n \underbrace{\int \binom{x}{n} d\mu}_{= b_n}.$$