# $L$-functions and the Riemann Hypothesis (DRAFT)

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# INTRODUCTION

These are notes for a set of lectures on "$L$-functions and the Riemann Hypothesis" at the 2018 CTNT summer school. It describes basic properties of Dirichlet $L$-functions, with the Riemann zeta-function as an important special case. Properties we consider include special values, analytic continuation, and functional equation, and finally the Riemann Hypothesis for these $L$-functions: what it says, how it can be tested numerically, and some of its applications.

For prerequisites, the reader is assumed to have already taken courses in complex analysis and abstract algebra. Some topics that should be familiar from complex analysis include: analytic and meromorphic functions, the idea of analytic continuation, and the residue theorem. From algebra the reader should know about homomorphisms and basic properties of rings.

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# CHAPTER 1

# INTRODUCTION TO THE ZETA-FUNCTION AND DIRICHLET $L$-FUNCTIONS

## 1.1 The Riemann Zeta-function

The series

$$\sum_{n\geqslant 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots,$$

converges absolutely when $\mathrm{Re}(s) > 1$ since

$$\sum_{n\geqslant 1} \left|\frac{1}{n^s}\right| = \frac{1}{n^{\mathrm{Re}(s)}} < \infty$$

on account of the convergence of $\int_1^\infty dx/x^{\mathrm{Re}(s)}$ when $\mathrm{Re}(s) > 1$. (Recall $n^s = e^{s\log n}$, so $|n^s| = n^{\mathrm{Re}(s)}$.) Its behavior as a function of a complex variable was first studied by Riemann (1859), so the series is named after him and uses his notation.

**Definition 1.1.** The *Riemann zeta-function* is

$$\zeta(s) := \sum_{n \geqslant 1} \frac{1}{n^s}$$

when $\mathrm{Re}(s) > 1$.

There are a host of similar functions, also called zeta-functions, introduced by mathematicians such as Dedekind, Epstein, Hasse, Hurwitz, Selberg, and Weil. Unless indicated otherwise, we will often write "the zeta-function" instead of "the Riemann zeta-function."

On the half-plane $\mathrm{Re}(s) > 1$, the series $\zeta(s)$ converges uniformly on compact subsets, and this implies $\zeta(s)$ is holomorphic on this region.

In addition to the series representation for $\zeta(s)$, there is a product representation involving a product over prime numbers: for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s}.$$

This product can be seen by expanding each factor $1/(1-1/p^s)$ into a geometric series and multiplying out all the series (the manipulations can be justified when $\mathrm{Re}(s) > 1$). Because the product for $\zeta(s)$ was first found by Euler, it is called the *Euler product* for $\zeta(s)$. The Euler product suggests there should be relations between $\zeta(s)$ and prime numbers, as indeed there are. The reason Riemann wrote his paper on the zeta-function was to sketch out a program for proving the Prime Number Theorem, which concerns the prime-counting function

$$\pi(x) = \#\{p \leqslant x : p \text{ prime}\}.$$

**Theorem 1.2 (Prime Number Theorem).** *As $x \to \infty$ we have*

$$\pi(x) \sim \frac{x}{\log x}, \tag{1.1}$$

*which means the two sides have ratio tending to* 1.

This theorem was conjectured by Gauss and Legendre independently in the early 19th century and was first proved by Hadamard and de la Vallée Poussin independently in 1896 using the zeta-function and the ideas about it proposed by Riemann. Eventually it was shown that the Prime Number Theorem is equivalent to the nonvanishing of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$. Another version of the Prime Number Theorem is

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}.$$

This is equivalent to (1.1) because the integral grows in the same way as $x/\log x$ when $x \to \infty$ (take the limit of their ratio using L'Hospital's rule). The integral on the right side is called the logarithmic integral of $x$ and denoted $\mathrm{Li}(x)$.

The series for $\zeta(s)$ diverges as $s \to 1^+$, and Riemann discovered how to analytically continue $\zeta(s)$ around the point $s = 1$, by moving through the upper or lower half-planes into regions where the series that originally defines $\zeta(s)$ in Definition 1.1 no longer converges. Specifically, multiply $\zeta(s)$ by a power of $\pi$ and a value of the $\Gamma$-function to define

$$\mathrm{Z}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

(See Section 2.3 for basic properties of the Gamma function $\Gamma(s)$, including the location of its poles.) Although initially defined only for $s$ with real part $> 1$, by using some ideas from Fourier analysis (the Poisson summation formula), $\mathrm{Z}(s)$ can be rewritten in the form

$$\mathrm{Z}(s) = \int_1^\infty \omega(x)(x^{s/2} + x^{(1-s)/2})\,\frac{dx}{x} - \frac{1}{s(1-s)}, \tag{1.2}$$

where $\omega(x)$ is the rapidly converging exponential series $\sum_{n\geqslant 1} e^{-\pi n^2 x}$ and this integral converges (and is holomorphic) at every $s \in \mathbf{C}$. So the right side of (1.2) extends $\mathrm{Z}(s)$ to a holomorphic function on all of $\mathbf{C}$, except for simple poles at $s = 0$ and $s = 1$ from the term $\dfrac{1}{s(1-s)}$. The equation

$$\zeta(s) = \frac{\pi^{s/2}\,\mathrm{Z}(s)}{\Gamma(s/2)}$$

now extends the zeta-function to all of $\mathbf{C}$, except for a simple pole at $s = 1$. (The simple poles of $\mathrm{Z}(s)$ and $\Gamma(s/2)$ at $s = 0$ cancel, so $\zeta(s)$ is holomorphic and nonvanishing at $s = 0$.) Because of the symmetry of (1.2) in $s$ and $1 - s$, we have

$$\mathrm{Z}(1-s) = \mathrm{Z}(s),$$

which is called the *functional equation*. We will derive the analytic continuation and functional equation for the zeta-function in Section 4.1.[1]

The functional equation can be rewritten as

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s), \tag{1.3}$$

a formula that expresses $\zeta(1-s)$ in terms of $\zeta(s)$, but here the symmetry in $s$ and $1-s$ is broken.

As we'll see later, it is easy to show that $\zeta(s)$ has simple zeros at the negative even integers and all its other zeros lie in $\{s : 0 \leqslant \mathrm{Re}(s) \leqslant 1\}$, which is called the *critical strip*. A more precise claim about the location of these remaining zeros is the famous

Riemann Hypothesis (RH): The zeros of $\zeta(s)$ with real part between 0 and 1 all lie on the vertical line $\mathrm{Re}(s) = 1/2$.

The vertical line $\mathrm{Re}(s) = 1/2$ is special for the zeta-function, being the line of symmetry for the transposition $s \mapsto 1-s$ in the functional equation. It is called the *critical line*.

Error estimates on the difference $|\pi(x) - \mathrm{Li}(x)|$ are closely related to the Riemann Hypothesis: RH is equivalent to the bound

$$|\pi(x) - \mathrm{Li}(x)| \leqslant C\sqrt{x}(\log x)^2.$$

for a constant $C > 0$. The exponent in the factor $\sqrt{x} = x^{1/2}$ comes from the "1/2" defining the critical line. **Note**: It is *false* that $|\pi(x) - x/\log x| \leqslant C\sqrt{x}(\log x)^2$, so although $x/\log x$ and $\mathrm{Li}(x)$ grow in the same way, there are not always interchangeable.

Exercises for Section 1.1

1. Write the letter $\zeta$ until you can do it easily and correctly (at least 25 times).

[1]To remove the two poles of $\mathrm{Z}(s)$ while retaining a functional equation, $s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is a common substitute for $\mathrm{Z}(s)$, and many authors in fact designate this product as $\mathrm{Z}(s)$. For largely historical reasons, an additional factor of 1/2 is sometimes included as well.

2. Show that $\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$ by verifying that

$$\lim_{x\to\infty} \frac{\int_2^x \frac{dt}{\log t}}{x/\log x} = 1.$$

3. a) Let $p_n$ denote the $n$th prime number, so $p_1 = 2$ and $p_4 = 7$. Use the Prime Number Theorem to show

$$n \sim \frac{p_n}{\log(p_n)}$$

as $n \to \infty$.

b) Deduce from the Prime Number Theorem that

$$\log(\pi(x)) \sim \log x$$

as $x \to \infty$. Then show $\log(p_n) \sim \log n$, so by part a) we get an asymptotic formula for the $n$th prime:

$$p_n \sim n \log n.$$

c) Show that the asymptotic relation $p_n \sim n \log n$ implies the Prime Number Theorem. (Hint: For $x \geqslant 2$, let $n = \pi(x)$, so $p_n \leqslant x < p_{n+1}$.)

d) Let $c > 0$ be a positive number. Use part b) to show

$$\lim_{n\to\infty} \frac{p_{[cn]}}{p_n} = c.$$

Here $[y]$ denotes the greatest integer $\leqslant y$. This formula shows that the ratios of the primes are dense in the positive reals, in a somewhat constructive manner.

e) The convergence of the limit in part d) is *very* slow. To see this numerically, use a computer algebra package to compute $p_{[\sqrt{2}n]}/p_n$ until you find some $n$ for which this ratio approximates $\sqrt{2} = 1.414\ldots$ correctly to two decimal places.

## 1.2 Dirichlet $L$-functions

At the same time that he conjectured the Prime Number Theorem, Legendre also conjectured that any arithmetic progression $a, a+2m, a+3m, \ldots$ contains infinitely many primes when $a$ and $m$ share no common factors. (There will be at most a finite number of primes in this progression if $a$ and $m$ have a common factor $> 1$.) This conjecture was first established by Dirichlet.

**Theorem 1.3 (Dirichlet, 1837).** *For integers $a, m \geqslant 1$ with no common factor, there are infinitely many primes in the arithmetic progression $a, a+m, a+2m, a+3m, \ldots$.*

For example, there are two arithmetic progressions with $m = 4$ consisting of numbers relatively prime to 4,

$$1, 5, 9, 13, 17, 21, 25, \ldots \qquad 3, 7, 11, 15, 19, 23, 27, \ldots,$$

and each contains infinitely many primes.

Dirichlet's proof of Theorem 1.3 uses a modification of the zeta-function that depends on a group homomorphism $\chi\colon (\mathbf{Z}/m\mathbf{Z})^\times \to S^1$, where $S^1$ is the set of complex numbers with absolute value 1.

**Definition 1.4.** A group homomorphism $\chi\colon (\mathbf{Z}/m\mathbf{Z})^\times$ is called a *Dirichlet character*, or a *Dirichlet character mod $m$* when we want to specify the modulus.

**Example 1.5.** The character $\chi_4$ on $(\mathbf{Z}/4\mathbf{Z})^\times$ is defined by the rule

$$\chi_4(a \bmod 4) = \begin{cases} 1, & \text{if } a \equiv 1 \bmod 4, \\ -1, & \text{if } a \equiv 3 \bmod 4. \end{cases}$$

A Dirichlet character mod $m$ is a multiplicative function defined on the integers prime to $m$. If $\chi$ is identically 1, we call $\chi$ a *trivial* Dirichlet character. There is a trivial Dirichlet character $\mathbf{1}_m$ for each modulus $m$. Consider $\chi$ as a function of period $m$ on all integers, where we define $\chi(n) = 0$ for integers $n$ having a factor in common with $m$. As a function on $\mathbf{Z}$, $\chi$ remains fully multiplicative: $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$ for all integers $n_1, n_2$. With $\chi$ defined for all integers, we make the following definition. In the case of the trivial character,

$$\mathbf{1}_m(n) = \begin{cases} 1, & \text{if } (n, m) = 1, \\ 0, & \text{if } (n, m) > 1. \end{cases}$$

**Definition 1.6.** For a Dirichlet character $\chi$, the *Dirichlet $L$-function*[2] or *Dirichlet $L$-series* of $\chi$ is

$$L(s,\chi) := \sum_{n\geqslant 1} \frac{\chi(n)}{n^s}.$$

**Example 1.7.** The $L$-function of $\chi_4$ is

$$L(s,\chi_4) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{9^s} + \cdots$$

when $\mathrm{Re}(s) > 1$, with alternating signs in the numerators and powers of odd numbers in the denominators.

The series for $L(s,\chi)$ converges absolutely when $\mathrm{Re}(s) > 1$, and uniformly on compact subsets so $L(s,\chi)$ is analytic when $\mathrm{Re}(s) > 1$. The multiplicativity of $\chi$ implies Dirichlet $L$-functions have Euler products: for $\mathrm{Re}(s) > 1$,

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)/p^s}.$$

Unlike the zeta-function, which has a pole at $s = 1$, $L$-functions of nontrivial characters are not problematic at 1. In fact, the *series* defining $L(s,\chi)$ when $\chi$ is a nontrivial character actually converges uniformly on compact subsets of the larger domain $\mathrm{Re}(s) > 0$.

**Example 1.8.** The series $L(s,\chi_4)$ converges for real $s > 0$ by the alternating series test. That there is convergence for complex $s$ with $\mathrm{Re}(s) > 0$ requires more work.

From the uniform convergence on compact subsets, $L(s,\chi)$ is analytic on $\mathrm{Re}(s) > 0$ but the series does not converge absolutely when $0 < \mathrm{Re}(s) \leqslant 1$. Whether or not the Euler product for $L(s,\chi)$ makes sense even on the line $\mathrm{Re}(s) = 1$ is very hard, and anywhere to the left of this line would amount to making progress on a Riemann Hypothesis for $L(s,\chi)$, which we will introduce below.

For nontrivial Dirichlet characters $\chi$, there is an analytic continuation of $L(s,\chi)$ to the whole complex plane; it has no pole, unlike for $\zeta(s)$. As we will

[2] It is not known why Dirichlet denoted his functions with an $L$. Perhaps he chose $L$ for Legendre. Or the reason may be alphabetical. Just before $L$-functions are introduced in [?], there are certain functions $G$ and $H$, and the letters $I, J$, and $K$ may not have seemed appropriate labels for a function. Both $L(s,\chi)$ and $L(\chi,s)$ are common notations, although neither is due to Dirichlet; he simply wrote different $L$-functions as $L_0, L_1, L_2, \ldots$.

see in Section **??**, if $\chi$ is nontrivial and satisfies a technical condition called primitivity, then $L(s,\chi)$ admits a functional equation relating $L(s,\chi)$ and $L(1-s,\overline{\chi})$, where $\overline{\chi}$ is the complex conjugate Dirichlet character to $\chi$. More precisely, multiplication of $L(s,\chi)$ by a suitable exponential and Gamma function produces an entire function $\Lambda(s,\chi)$ that satisfies the functional equation

$$\Lambda(s,\chi) = W(\chi)\Lambda(1-s,\overline{\chi}),$$

where $W(\chi)$ is a complex number of absolute value 1.

**Example 1.9.** For the character $\chi_4$,

$$\Lambda(s,\chi_4) := \left(\frac{4}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi_4),$$

which turns out to be an entire function with the same value at $s$ and $1-s$:

$$\Lambda(1-s,\chi_4) = \Lambda(s,\chi_4).$$

Here $\overline{\chi_4} = \chi_4$ since $\chi_4$ is real-valued, and $W(\chi_4) = 1$.

The zeros of $L(s,\chi)$ include simple zeros at either the negative odd integers or the negative even integers (together with 0), and all other zeros have real part between 0 and 1. Analogously to $\zeta(s)$, for each $L(s,\chi)$ there is a

Generalized Riemann Hypothesis (GRH): The zeros of $L(s,\chi)$ with real part between 0 and 1 all lie on the vertical line $\mathrm{Re}(s) = 1/2$.

Exercises for Section 1.2

1. For an integer $n$, set

$$\chi_3(n) = \begin{cases} 1, & \text{if } n \equiv 1 \bmod 3, \\ -1, & \text{if } n \equiv 2 \bmod 3, \\ 0, & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

   Check $\chi_3$ is a Dirichlet character and show $L(s,\chi_3)$ converges for $s > 0$.

2. Show there are four Dirichlet characters mod 5. Describe them explicitly.

# CHAPTER 2

# SOME TOPICS FROM ANALYSIS

In this chapter we review concepts from analysis that are used in the proofs of analytic properties of $\zeta(s)$ and $L(s, \chi)$.

## 2.1 Rates of Growth

The concept of "rate of growth" involves a comparison through ratios, not differences. We briefly discuss the three types of growth estimates that will be needed: asymptotic estimates, $O$-estimates, and $o$-estimates.

All functions considered in this section will take real values.

When $f(x)/g(x) \to 1$ as the real variable $x \to \infty$, we write $f \sim g$ and say $f$ is *asymptotic* to $g$. For example, $3x^2 - 2x \sim 3x^2$ and $[x] \sim x$. However, $e^{[x]} \not\sim e^x$. (This is a good example of nonasymptotic functions to remember.) Since rates of growth are linked to derivatives, L'Hôpital's rule can be used to prove asymptotic relations if the functions are differentiable.

The asymptotic relation makes sense also for functions of an integer variable $n$ (*i.e.*, for sequences of real numbers), taking a limit as $n \to \infty$ through the positive integers. It is easy to see that the asymptotic relation respects addition for sequences of positive numbers: if $a_n \sim b_n$ and $c_n \sim d_n$, then $a_n + c_n \sim b_n + d_n$.

However, you can't always subtract. For example, $n^2+n \sim n^2+2n$ and $n^2 \sim n^2$, but $(n^2+n) - n^2 \not\sim (n^2+2n) - n^2$.

In the course of making estimates, the relation $|f(x)| \leqslant Cg(x)$ comes up often. Here $C$ is a constant that does not depend on the real variable $x$, but may depend on other parameters in the problem. We write $f(x) = O(g(x))$ (read: $f$ is big-Oh of $g$) if $g(x) \geqslant 0$ and $|f(x)| \leqslant Cg(x)$ for some constant $C$ and all large $x$. So $f(x) = O(g(x))$ means $f(x)$ grows no faster than $g(x)$, up to a constant multiple. (An alternate notation for $f(x) = O(g(x))$ is $f(x) \ll g(x)$. We won't use this.) The reason for insisting $g(x) \geqslant 0$ is that we want the relations $f_1(x) = O(g_1(x))$ and $f_2(x) = O(g_2(x))$ to imply $f_1(x) + f_2(x) = O(g_1(x) + g_2(x))$; this would be false if we defined the big-Oh notation as $|f(x)| \leqslant C|g(x)|$.

As examples, $10x^2 - 17x + 5 = O(x^2)$ and $12 \sin x = O(1)$. Saying $f(x) = O(1)$ is another way of saying $f$ is a bounded function (as $x \to \infty$), and the $O(1)$ notation is useful when we are not concerned with the precise bounding constant. For that matter, the convenience of the $O$-notation in general is to carry estimates through a series of equations (really, inequalities) when the constants that bound relative growth rates are not important.

If $f(x) = O(x^a)$, then $f(x) = O(x^{a+\delta})$ for all $\delta > 0$, but the converse is false; $\log x = O(x^\delta)$ for all $\delta > 0$ but $\log x \neq O(1)$. The constant implicit in the estimate $\log x = O(x^\delta)$ depends on $\delta$, and this fact could be indicated by writing $\log x = O_\delta(x^\delta)$.

Estimates on the growth of a difference $f(x+1) - f(x)$ can often be discovered by using the Mean Value Theorem or looking at a Taylor expansion. For instance, $\log(x+1) - \log x$ is between $1/x$ and $1/(x+1)$ by the Mean Value Theorem, so $\log(x+1) = \log x + O(1/x)$. Alternatively, $\log(x+1) - \log x = \log(1 + 1/x)$ has a Taylor expansion in $1/x$ with constant term 0, so again $\log(x+1) = \log x + O(1/x)$.

If $f(x)/g(x) \to 0$ as $x \to \infty$, we'll say $f$ has a *strictly slower* rate of growth than $g$, and write $f = o(g)$ (read: $f$ is little-oh of $g$). For instance, $x^2 = o(x^3)$ (more generally, $x^a = o(x^b)$ if $0 < a < b$) and $\log x = o(x^\delta)$ for any $\delta > 0$. The equation $f = o(1)$ simply means $f(x) \to 0$ as $x \to \infty$. For example, we can write the limit relation $\sqrt{x^2+x} - x \to 1/2$ as

$$\sqrt{x^2+x} = x + \frac{1}{2} + o(1).$$

As with the asymptotic relation, the $O$ and $o$ relations apply to sequences as well as functions of a real variable.

It is useful to be aware of how to transfer between the relations $\sim, O$, and $o$. For example, if $f \sim g$ and $f, g > 0$, then $\log f = \log g + o(1)$. This is more accurate than saying $\log f \sim \log g$. (In general, if $f_1 = f_2 + o(1)$ then $f_1 \sim f_2$.) As another example, the relation $\log(x+1) = \log x + O(1/x)$ is more informative than either $\log(x+1) = \log x + o(1)$ or $\log(x+1) \sim \log x$. An even better estimate is $\log(x+1) = \log x + 1/x + O(1/x^2)$.

Note that equations involving growth estimates are usually not symmetric, *e.g.*, $x\log(x+1) + O(\sqrt{x}) = x\log x + o(x)$ but $x\log x + o(x) \neq x\log(x+1) + O(\sqrt{x})$. Equations using the $O$ and $o$ notation are always read left to right, not right to left. So they're not really equations so much as relations.

Since any sequence $a_n$ can be extended to a function $a(x)$ by setting $a(x) = a_{[x]}$, and many classical functions satisfy $f(x) \sim f([x])$ if they don't grow too quickly (a basic counterexample is $e^x$), we can study rates of growth of sequences as rates of growth of functions (of a real variable) and vice versa. Functions are often more convenient because the tools of analysis are available. For instance, it is convenient to describe partial sums using a real variable as upper bound: $\sum_{n\leqslant x} a_n$ means $\sum_{n\leqslant [x]} a_n$. In the context of partial sums, note that the relation $\sum_{n\leqslant x} a_n = o(x)$ just means that the sequence of averages $(a_1 + a_2 + \cdots + a_n)/n$ tends to 0.

That we have defined the relations $\sim, O$, and $o$ only as the variable $\to \infty$ is not essential. We can consider these relations as the variable tends towards a finite value, possibly from only one direction. Another example is probably the first nonobvious limit formula learned in calculus: $\sin\theta \sim \theta$ as $\theta \to 0$. The estimate at $\infty$ is completely different: $\sin\theta = o(\theta)$ as $\theta \to \infty$. To avoid confusion, a growth estimate should be accompanied by an indication of where the variable is tending, unless there is some convention made in advance. So we make a convention: in light of the way we will be using growth estimates, the variable in a growth estimate will $\to \infty$ unless indicated otherwise.

Exercises for Section 2.1

1. Show $\log x = o(x^\delta)$ for all $\delta > 0$.

2. Show $f(x) = O(x^\varepsilon)$ for all $\varepsilon > 0$ if and only if $f(x) = o(x^\varepsilon)$ for all $\varepsilon > 0$. (Here the $O-$ and $o-$symbols may depend on $\varepsilon$.)

3. Suppose $f$ is differentiable and the derivative $f'$ is "weakly increasing": $a \leqslant b$ implies $f'(a) \leqslant f'(b)$. If $f(x) \sim x^a$ for a positive constant $a$, show

$f'(x) \sim ax^{a-1}$. (Hint: Use $(f((1+\varepsilon)x)-f(x))/\varepsilon x$ as an approximation to $f'(x)$, and get a lower bound on the Newton quotient by the Mean Value Theorem. Divide by $x^{a-1}$, let $x \to \infty$, and then let $\varepsilon \to 0^+$.)

4. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences of positive numbers with $a_n \sim b_n$ and $c_n \sim d_n$. That is, $a_n/b_n \to 1$ and $c_n/d_n \to 1$ as $n \to \infty$. Show $a_n + c_n \sim b_n + d_n$.

5. Let $a_n, b_n$ be sequences of positive numbers such that $a_n \sim b_n$ and the series $\sum a_n$ and $\sum b_n$ both converge. Show the tails of these series are asymptotic: $\sum_{n \geqslant N} a_n \sim \sum_{n \geqslant N} b_n$ as $N \to \infty$.

6. a) For $r > 0$, show $\sum n^r z^n$ converges absolutely for $|z| < 1$.

   b) If $r$ is a nonnegative integer, show

$$\sum_{n \geqslant 1} n^r z^n = \frac{P_r(z)}{(1-z)^{r+1}},$$

   where $P_r(z)$ is a monic polynomial of degree $r$ with integer coefficients, constant term 0 (for $r > 0$), and $P_r(1) = r!$. In particular,

$$\sum_{n \geqslant 1} n^r y^n \sim \frac{r!}{(1-y)^{r+1}}$$

   as $y \to 1^-$.

   c) For $r > 0$, show the coefficients of $P_r(z)$ (except the constant term) are all positive.

7. Let $A = \{a_1, a_2, \dots\}$ be a set of positive real numbers (not necessarily integers) with $a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots$ and

$$\#\{a \in A : a \leqslant x\} \sim C \frac{x}{\log x}$$

   for a positive constant $C$.

   a) Show $a_n \sim (1/C) n \log n$.

   b) Show $\#\{a \in A : a \leqslant a_n\} \sim n$.

## 2.2 Infinite Series and Products

Given a sequence $\{z_1, z_2, \dots\}$ in $\mathbf{C}$, the series $\sum z_n = \sum_{n \geqslant 1} z_n$ is defined to be the limit of the partial sums $\sum_{n=1}^{N} z_n$, as $N \to \infty$. For a permutation $\pi$ of the positive integers, we can consider the rearranged series $\sum z_{\pi(n)}$. (In this section, $\pi(n)$ always represents a permuted value, not the prime counting function in the Prime Number Theorem.) Is there a difference between the series $\sum z_n$ and $\sum z_{\pi(n)}$ when both converge? Perhaps.

Consider the series

$$\sum_{n \geqslant 1} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2 = .693147\dots.$$

Write the $n$th term as $z_n$, so $z_n = (-1)^{n-1}/n$. Let's rearrange the terms, computing the sum in the following order:

$$z_1 + z_2 + z_4 + z_3 + z_6 + z_8 + z_5 + z_{10} + z_{12} + z_7 + \dots$$

So we place two even-indexed terms after an odd-indexed term. This sum looks like

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} + \dots \leqslant .452$$

The general term tends to 0 and from the sign changes the sum converges to a value less than $\log 2$. Combining $z_1$ and $z_2$, $z_3$ and $z_6$, $z_5$ and $z_{10}$, etc., this sum equals

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{7} + \cdots = \frac{1}{2} \log 2 = .346573\dots.$$

We've rearranged the terms from a series for $\log 2$ and obtained half the original value.

Until the early 19th century, the evaluation of infinite series was not troubled by rearrangement issues as above, since there wasn't a clear distinction between two issues: defining convergence of a series and computing a series. An infinite series needs a precise defining algorithm, such as taking a limit of partial sums via an enumeration of the addends, upon which other summation methods may or *may not* be comparable.

Recall that a series $\sum z_n$ of complex numbers is called *absolutely convergent* if the series $\sum |z_n|$ of absolute values of the terms converges. This condition is usually first seen in a calculus course, as a "convergence test." Indeed, the partial sum differences satisfy

$$\left|\sum_{n=M}^{N} z_n\right| \leqslant \sum_{n=M}^{N} |z_n|$$

and the right side tends to 0 as $M, N \to \infty$ if $\sum |z_n|$ converges, so the left side tends to 0 and therefore $\sum z_n$ converges.

Absolutely convergent series are ubiquitous. For example, a power series $\sum c_n z^n$ centered at the origin that converges at $z_0 \neq 0$ converges absolutely at every $z$ with $|z| < |z_0|$. (Proof: Let $r = |z/z_0| < 1$ and $|c_n z_0^n| \leqslant B$ for some bound $B$. Then $\sum |c_n z^n|$ is bounded by the geometric series $\sum Br^n < \infty$.) An analogous result applies to series centered at points other than the origin.

The behavior of absolutely convergent series is related to convergent series of nonnegative real numbers, and such series have very convenient properties, outlined in the following lemmas.

**Lemma 2.1.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers. If the partial sums $\sum_{n=1}^{N} a_n$ are bounded, then the series $\sum_{n\geqslant 1} a_n$ converges. Otherwise it diverges to $\infty$.*

*Proof.* The partial sums are an increasing sequence (perhaps not strictly increasing, since some $a_n$ may equal 0), so if they have an upper bound they converge, and if there is no upper bound they diverge to $\infty$. ∎

**Lemma 2.2 (Generalized Commutativity).** *Let $a_n \geqslant 0$ and assume the series $\sum_{n\geqslant 1} a_n$ converges, say to $S$. For every permutation $\pi$ of the index set, the series $\sum a_{\pi(n)}$ also converges to $S$.*

*Proof.* Choose $\varepsilon > 0$. For all large $N$, say $N \geqslant M$ (where $M$ depends on $\varepsilon$),

$$S - \varepsilon \leqslant \sum_{n=1}^{N} a_n \leqslant S + \varepsilon.$$

The permutation $\pi$ takes on all values $1, 2, \ldots, M$ among some initial segment of the positive integers, say

$$\{1, 2, \ldots, M\} \subset \{\pi(1), \pi(2), \ldots, \pi(K)\}$$

for some $K$. For $N \geqslant K$, the set $\{a_{\pi(1)}, \ldots, a_{\pi(N)}\}$ contains $\{a_1, \ldots, a_M\}$. Let

$J$ be the maximal value of $\pi(n)$ for $n \leqslant N$. So for $N \geqslant K$,

$$S - \varepsilon \leqslant a_1 + a_2 + \cdots + a_M \leqslant \sum_{n=1}^{N} a_{\pi(n)} \leqslant a_1 + a_2 + \cdots + a_J \leqslant S + \varepsilon.$$

So for every $\varepsilon$, $\sum_{n=1}^{N} a_{\pi(n)}$ is within $\varepsilon$ of $S$ for all large $N$. Therefore $\sum a_{\pi(n)} = S$. ∎

Because of Lemma 2.2, we can associate to a sequence $\{a_i\}$ of nonnegative real numbers indexed by a countable index set $I$ the series $\sum_{i \in I} a_i$, by which we mean the limit of partial sums for any enumeration of the terms. If it converges in one enumeration it converges in all others, to the same value.

We apply the idea of a series running over a general countable index set right away in the next lemma.

**Lemma 2.3 (Generalized Associativity).** *Let $\{a_i\}$ be a sequence of nonnegative real numbers with countable index set $I$. Let*

$$I = I_1 \cup I_2 \cup I_3 \cup \cdots$$

*be a partition of the index set. If the series $\sum_{i \in I} a_i$ converges, then so does each series*

$$s_j = \sum_{i \in I_j} a_i,$$

*and*

$$\sum_{i \in I} a_i = \sum_{j \geqslant 1} s_j = \sum_{j \geqslant 1} \left( \sum_{i \in I_j} a_i \right).$$

*Conversely, if each $s_j$ converges and the series $\sum_{j \geqslant 1} s_j$ converges, then the series $\sum_{i \in I} a_i$ converges to $\sum_{j \geqslant 1} s_j$.*

*Proof.* Exercise. ∎

The importance of absolutely convergent series is that they satisfy the above convenient properties of series of nonnegative numbers.

**Theorem 2.4.** *Let $a_i$ be a sequence of complex numbers. Assume $\sum |a_i|$ converges, i.e., $\sum a_i$ is absolutely convergent. Then Lemmas 2.2 and 2.3 apply to $\sum a_i$.*

*Proof.* Exercise. ∎

The definition of an absolutely convergent series of complex numbers makes sense with any countable indexing set, not only index set $\mathbf{Z}^+$. This is often technically convenient since many useful indexing sets are not the positive integers.

Theorem 2.4 is used to justify interchanging the order of a double summation $\sum_m \sum_n a_{mn}$, if it is absolutely convergent, *i.e.*, if $\sum_m \sum_n |a_{mn}|$ converges. The theorem justifies the rearrangements of certain double sums $\sum_p \sum_{k\geqslant 1} c_{p^k}$ over primes $p$ and positive integers $k$ into a single sum $\sum c_{p^k}$ over prime powers $p^k$ in their usual linear ordering: $\sum c_{p^k} := \lim_{x\to\infty} \sum_{p^k\leqslant x} c_{p^k}$. (Note: Since $k \geqslant 1$ here, there is no term corresponding to the prime power $1 = p^0$ for all $p$.)

We will want to work not only with series, but with infinite products. We generally work with infinite products only in cases where techniques related to absolutely convergent series play a role.

**Theorem 2.5.** *Let $\sum a_m$, $\sum b_n$ be two absolutely convergent series. The product of the sums of these two series is the sum of the absolutely convergent series $\sum a_m b_n$ over the index set $\mathbf{Z}^+ \times \mathbf{Z}^+$. More generally, a finite product of sums of absolutely convergent series is again the sum of an absolutely convergent series, whose terms are all the possible products of terms taken one from each of the original series.*

*Proof.* It suffices to treat the case of a product of two series.

Let $S = \sum a_m$, $T = \sum b_n$. Then

$$ST = \sum_n S b_n = \sum_n \left( \sum_m a_m b_n \right).$$

The terms $a_m b_n$ give an absolutely convergent series if we think of them as being indexed by the countable index set $I = \mathbf{Z}^+ \times \mathbf{Z}^+$. Partitioning this index set into its rows or its columns and using generalized associativity equates the double series $\sum_n(\sum_m a_m b_n)$ and $\sum_m(\sum_n a_m b_n)$ with $\sum_{(m,n)\in\mathbf{Z}^+\times\mathbf{Z}^+} a_m b_n$. ■

The following theorem contains most of what we shall need concerning relations between infinite products and infinite series. Although we will review complex logarithms in the following section, for now we take for granted the basic property that the series $\log(1-z) := \sum_{m\geqslant 1} z^m/m$ for $|z| < 1$ is a right inverse to the exponential function: $\exp(\log(1-z)) = 1 - z$.

**Theorem 2.6.** *Let $\{z_n\}$ be complex numbers with $|z_n| \leqslant 1-\varepsilon$ for some positive $\varepsilon$ (which is independent of $n$) and $\sum |z_n|$ convergent.*

*a) The infinite product* $\prod_{n\geqslant 1}\frac{1}{1-z_n} = \lim_{N\to\infty}\prod_{n=1}^{N}\frac{1}{1-z_n}$ *converges to a nonzero number.*

*b) This infinite product satisfies generalized commutativity (i.e., every rearrangement of factors gives the same product) and generalized associativity for products.*

*c) The product* $\prod_{n\geqslant 1}\frac{1}{1-z_n} = \prod_{n\geqslant 1}(1+z_n+z_n^2+\dots)$ *has a series expansion by collecting terms in the expected manner:*

$$\prod_{n\geqslant 1}\frac{1}{1-z_n} = 1+\sum_{r\geqslant 1}\sum_{\substack{k_1,\dots,k_r\geqslant 1\\ 1<i_1<\cdots<i_r}} z_{i_1}^{k_1}\cdots z_{i_r}^{k_r},$$

*and this series is absolutely convergent.*

*Proof.* We consider the naive logarithm of the infinite product, which is, by definition,

$$-\sum_{n\geqslant 1}\log(1-z_n) := \sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{z_n^m}{m}.$$

(This equation is a definition of the left side; there is no claim that $\log(zw) = \log z + \log w$.)

Since

$$\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{|z_n|^m}{m} \leqslant \sum_{n\geqslant 1}\sum_{m\geqslant 1}|z_n|^m = \sum_{n\geqslant 1}\frac{|z_n|}{1-|z_n|} \leqslant \frac{1}{\varepsilon}\sum_{n\geqslant 1}|z_n| < \infty,$$

the doubly indexed sequence $\{z_n^m/m\}$ is absolutely convergent, so it satisfies generalized commutativity and associativity for series. By continuity of the exponential,

$$\exp\left(\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{z_n^m}{m}\right) = \prod_{n\geqslant 1}\exp\left(\sum_{m\geqslant 1}\frac{z_n^m}{m}\right) = \prod_{n\geqslant 1}\frac{1}{1-z_n},$$

so the product converges to a nonzero number (since it's a value of the exponential) and this product satisfies generalized commutativity and associativity since the double sum in the exponential does. We have taken care of a) and b).

For c), we use absolute convergence of each $\sum_{j\geqslant 1} z_n^j$ to write the product

$P_N := \prod_{n=1}^{N} \sum_{j \geqslant 0} z_n^j$ as

$$\sum_{j_1,\ldots,j_N \geqslant 0} z_1^{j_1} \cdots z_N^{j_N} = 1 + \sum_{r=1}^{N} \sum_{\substack{k_1,\ldots,k_r \geqslant 1 \\ 1 \leqslant i_1 < \cdots < i_r \leqslant N}} z_{i_1}^{k_1} \cdots z_{i_r}^{k_r}.$$

This equation follows from generalized associativity. Now let $N \to \infty$ to get convergence of the series in part c).

Replace each $z_n$ with its absolute value $|z_n|$:

$$\begin{aligned} \prod_{n \geqslant 1} \frac{1}{1 - |z_n|} &\geqslant \prod_{n=1}^{N} \frac{1}{1 - |z_n|} \\ &= 1 + \sum_{r=1}^{N} \sum_{\substack{k_1,\ldots,k_r \geqslant 1 \\ 1 \leqslant i_1 < \cdots < i_r \leqslant N}} |z_{i_1}|^{k_1} \cdots |z_{i_r}|^{k_r}. \end{aligned}$$

Let $N \to \infty$ to see the series formed by $\{z_{i_1}^{k_1} \cdots z_{i_r}^{k_r}\}$ is absolutely convergent. ■

Exercises for Section 2.2

1. Prove Lemma 2.3 and Theorem 2.4.

2. Let $\{c_{p^k}\}$ be a sequence of complex numbers indexed by the prime powers, with $c_1 = 1$. For a positive integer $n = p_1^{e_1} \cdots p_r^{e_r}$ factored into distinct primes, set

$$d_n = c_{{p_1}^{e_1}} \cdots c_{{p_r}^{e_r}}.$$

   For instance, $d_1 = 1$. If $\sum d_n$ converges absolutely, show each series $\sum_{k \geqslant 0} c_{p^k}$ converges absolutely and

$$\prod_p \left( \sum_{k \geqslant 0} c_{p^k} \right) = \sum_{n \geqslant 1} d_n.$$

## 2.3 Complex Analysis

In this section we recall some results from complex analysis. It is assumed that the reader has already had a first course in complex analysis, so is familiar with terms like analytic (or holomorphic), meromorphic, pole, and residue. We will recall some facts about analytic functions and singularities, and then focus specifically on logarithms and the Gamma function.

When we integrate along a contour $\gamma$, we assume $\gamma$ is "nice," say a union of piecewise differentiable curves. If the endpoints of $\gamma$ coincide, we call $\gamma$ a loop, and if $\gamma$ does not cross itself (except perhaps at the endpoints) we call $\gamma$ simple.

Let $\Omega$ be an open set in $\mathbf{C}$ and $f\colon \Omega \to \mathbf{C}$ be a continuous function. The first miracle of complex analysis is that the property of $f$ being analytic (also called holomorphic) can be described in several different ways: complex differentiability of $f$ at each point in $\Omega$, local power series expansions for $f$ at each point in $\Omega$, or that $\int_\gamma f(z)\,dz = 0$ for any (or any sufficiently small) contractible loop $\gamma$ in $\Omega$. That the real and imaginary parts of $f$ satisfy the Cauchy–Riemann equations is another formulation of analyticity, important for links between complex analysis and partial differential equations. Notice in particular that the notion of analyticity is a *local* one, such as in the characterization by local power series expansions. Through the Cauchy integral formula these local conditions lead to global consequences, like the following.

**Theorem 2.7.** *Let $f$ be an analytic function on $D(a,r)$, the open disc around a point $a$ with radius $r$. Then the power series for $f$ at the center $a$ converges on $D(a,r)$.*

*Proof.* Let $r' < r$, and let $\gamma_{r'}(t) = a + r'e^{it}$ for $t \in [0, 2\pi]$ be the circular path around $a$ of radius $r'$, traversed once counterclockwise. For $z \in D(a, r')$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(w)}{w-z}\,dw.$$

For $w$ lying on $\gamma_{r'}$ (*i.e.*, $|w-a| = r'$) we have $|z-a| < |w-a|$ and so

$$\frac{1}{w-z} = \frac{1}{w-a}\cdot\frac{1}{1-(z-a)/(w-a)} = \sum_{n\geqslant 0}\frac{(z-a)^n}{(w-a)^{n+1}},$$

with the series converging uniformly in $w$. Multiplying both sides by $f(w)$ and integrating along $\gamma_{r'}$ we may interchange the sum and integral (by uniform convergence) and get

$$f(z) = \sum_{n\geqslant 0} c_n (z-a)^n, \tag{2.1}$$

where

$$c_n = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(w)}{(w-a)^{n+1}}\,dw.$$

So (2.1) is a power series expansion for $f$ around $a$ that converges on the open disc $D(a, r')$. Thus $c_n = f^{(n)}(a)/n!$, so it is independent of $r'$. Now let $r' \to r^-$. The series (2.1) doesn't change, so it applies for all $z \in D(a, r)$. ∎

This theorem is false for $1/(1 + x^2)$ as a real function, where we replace balls with intervals. Indeed, this function has local power series expansions at all points of $\mathbf{R}$ but its power series at, say, the origin does not have an infinite radius of convergence.

**Corollary 2.8.** *Let $\Omega$ be an open set in the plane and $f$ be an analytic function on $\Omega$. If a disc $D$ with radius $r$ is in $\Omega$, the power series of $f$ at the center of $D$ has radius of convergence at least $r$.*

**Theorem 2.9.** *Let $\Omega$ be open in the plane, $f_n$ a sequence of analytic functions on $\Omega$ that converges uniformly to $f$ on each compact subset of $\Omega$. Then $f$ is analytic and $f_n'$ converges uniformly to $f'$ on each compact subset.*

*Proof.* Choose $a \in \Omega$. Let $\overline{D} \subset \Omega$ be a closed disc of radius $R > 0$ containing $a$ in its interior. So

$$\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-a}\,dz = f_n(a),$$

where $\gamma$ traverses the boundary of $\overline{D}$ once counterclockwise. Since $f_n \to f$ uniformly on $\overline{D}$, $f$ is continuous on $\overline{D}$, so $f(z)/(z-a)$ is integrable along $\gamma$. Letting the maximum value of a function $g$ on $\overline{D}$ be written $||g||_{\overline{D}}$,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{z-a}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz\right| &\leqslant \frac{1}{2\pi}||f_n - f||_{\overline{D}}\left|\int_\gamma \frac{dz}{z-a}\right| \\ &= ||f_n - f||_{\overline{D}} \\ &\to 0.\end{aligned}$$

So $\frac{1}{2\pi i}\int_\gamma (f(z)/(z-a))\,dz = \lim_{n\to\infty} f_n(a) = f(a)$. Since $a$ was arbitrary, $f$ is analytic. To show $f_n' \to f'$ uniformly on compact subsets of $\Omega$, it suffices to work with closed discs. Let $\overline{D}$ be a closed disc in $\Omega$ with radius $R > 0$. Choose $a$ in the interior of $\overline{D}$. Then

$$f_n'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(z)}{(z-a)^2}\,dz, \quad f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz.$$

So $|f_n'(a) - f'(a)| \leqslant ||f_n - f||_{\overline{D}}/R \to 0$. ∎

While all analytic functions are (locally) expressible as series, some analytic functions are convenient to introduce in other ways, such as by an integral depending on a parameter. Here is a basic theorem in this direction.

**Theorem 2.10.** *Let $f(x, s)$ be a continuous function, where the real variable $x$ runs over an interval from $a$ to $b$ and $s$ varies over an open set $U$ in the plane. Suppose, for each $x$, that $f(x, s)$ is analytic in $s$. Then $F(s) := \int_a^b f(x, s)\,dx$ is analytic in $s$ and $F$ may be differentiated under the integral sign: $F'(s) = \int_a^b (\partial_2 f)(x, s)\,dx$.*

*Proof.* See [**?**, Chap. XV, Lemma 1.1]. ∎

**Theorem 2.11.** *Let $f\colon \Omega \to \mathbf{C}$ be analytic on a region $\Omega$, with $\overline{\Omega} = \{\overline{s} : s \in \Omega\}$ the conjugate region. Then $f^*(s) := \overline{f(\overline{s})}$ is analytic on $\overline{\Omega}$, with derivative $\overline{f'(\overline{s})}$.*

In practice this will be applied to self-conjugate regions, such as right half-planes.

*Proof.* Using local series expansions, the operation $f \mapsto f^*$ transforms a power series $\sum c_n (s-a)^n$ into $\sum \overline{c}_n (s - \overline{a})^n$. So analyticity of $f^*$ and the formula for its derivative are obvious. A proof relying instead on the definition of the complex derivative is possible (Exercise 2.3.2). ∎

Although analyticity is defined as a property of functions on open sets, it is convenient to have the notion available for functions on any set (especially closed sets). A function defined on any set $A$ in the plane is called analytic if it has a local power series expansion around each point of $A$, or equivalently if it is the restriction to $A$ of an analytic function defined on an open set containing $A$.

For example, an analytic function at a point is simply an analytic function on some open ball around the point. Since any open set containing a closed ball $B := \{z : |z| \leqslant r\}$ will contain some open ball $\{z : |z| < r + \varepsilon\}$ (this is because $B$ is compact), an analytic function on a closed ball is the restriction of an analytic function on some larger open ball (not merely larger open set). In contrast, half-planes are not compact, so an open set $\Omega$ containing a closed half-plane $H := \{z : \mathrm{Re}(z) \geqslant \sigma_0\}$ does *not* have to contain any open half-plane $\{z : \mathrm{Re}(z) > \sigma_0 - \varepsilon\}$. Indeed, the complement of $H$ in $\Omega$ could become arbitrarily thin as we move far away from the real axis. So a function that is analytic on a closed half-plane is not guaranteed to be the restriction of an analytic function on a larger open half-plane.

The reader should already be familiar with the two standard types of singularities for an analytic function $f$: poles and essential singularities. These concepts apply to isolated points, namely points $a$ such that $f$ is analytic on a punctured disc $0 < |z - a| < r$. Whether $a$ is a pole or an essential singularity of $f$ can be characterized by either the behavior of $|f(z)|$ as $z \to a$ or by the Laurent series expansion for $f$ at $a$. In both cases $|f|$ is necessarily unbounded near $a$, by the following result of Riemann.

**Theorem 2.12 (Riemann's Removable Singularity Theorem).** *Let $f$ be holomorphic on an open set around the point $a$, except possibly at $a$. If $f$ is bounded near $a$, then $f$ extends to an analytic function at $a$.*

The converse of the theorem is trivial.

*Proof.* If $f(z)$ is bounded near $a$, then the function

$$g(z) = \begin{cases} (z-a)^2 f(z), & \text{if } z \neq a, \\ 0, & \text{if } z = a, \end{cases}$$

is certainly analytic in a punctured disc around $a$ and is continuous at $a$, with $g(a) = 0$. It is also differentiable at $a$: for $z \neq a$,

$$\frac{g(z) - g(a)}{z - a} = \frac{g(z)}{z - a} = (z - a)f(z) \to 0$$

as $z \to a$. Therefore $g$ is analytic in a neighborhood of $a$ with $g(a) = 0$ and $g'(a) = 0$. The power series expansion of $g$ around $a$ therefore begins

$$g(z) = c_2(z-a)^2 + c_3(z-a)^3 + c_4(z-a)^4 + \dots,$$

which shows $f(z)$ extends to an analytic function at $a$, with power series $c_2 + c_3(z-a) + c_4(z-a)^2 + \dots$. ∎

It is crucial that we assume analyticity on a punctured neighborhood of $a$ in the theorem. If $a$ is not isolated in this way, the theorem is false. Of course, to make sense of this we need to have a notion of singularity that does not apply only to isolated points.

**Definition 2.13.** Let $f\colon \Omega \to \mathbf{C}$ be analytic on an open set $\Omega$, $a \in \partial\Omega$ a point on the boundary of $\Omega$. We call $a$ an *analytic singularity* of $f$ if there is no extension of $f$ to an analytic function at $a$.

In other words, $a$ is an analytic singularity of $f$ if there is no power series centered at $a$ that on the overlap of its disc of convergence and $\Omega$ coincides with $f$. (That is, $f$ does not admit an analytic continuation to a neighborhood of $a$.) This is singular behavior from the viewpoint of complex analysis, but it does *not* mean $f$ has to behave pathologically near $a$ from the viewpoint of topology or real analysis. For instance, there are analytic functions on the open unit disc that extend continuously (even smoothly in the sense of infinite real-differentiability) to the unit circle, but at none of these boundary points is there a local power series expansion in a complex variable for the extended function. See [**?**, pp. 252–253] for an example.

Next we discuss some special functions: complex exponentials, logarithms, square roots, and the Gamma function.

The complex exponential is defined as $e^s := \sum_{n \geqslant 0} s^n/n!$ for all $s \in \mathbf{C}$.

**Definition 2.14.** For $u > 0$ and $s \in \mathbf{C}$, $u^s := e^{s \log u}$, where $\log u$ is the usual real logarithm of $u$.

Note that $u^{it} = e^{it \log u} = \cos(t \log u) + i \sin(t \log u)$, so $|u^{it}| = 1$. Clearly $|u^s| = u^{\mathrm{Re}(s)}$, so $u^s = 1 \Leftrightarrow s \in (2\pi i/\log u)\mathbf{Z}$. If $\mathrm{Re}(s) \geqslant 0$, then $\lim_{u \to 0^+} u^s = 0$.

Since the exponential function is not injective on $\mathbf{C}$, some care is needed to define logarithms. The usual remedy in a first course in complex analysis is to consider the *slit plane*

$$\{z \in \mathbf{C} : z \notin (-\infty, 0]\}$$

obtained by removing the negative real axis and 0. We can uniquely write each element of this slit plane in the form $z = re^{i\theta}$ with $-\pi < \theta < \pi$ and we define

$$\mathrm{Log}\, z := \log r + i\theta. \tag{2.2}$$

Here $\log r$ is the usual real logarithm. This function $\mathrm{Log}\, z$ is called the *principal value logarithm*, and specializes to the usual logarithm on the positive reals. The justification for calling $\mathrm{Log}\, z$ a logarithm is that it is a right inverse to the exponential function:

$$e^{\mathrm{Log}\, z} = e^{\log r + i\theta} = re^{i\theta} = z.$$

Notice that $\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ in general. For instance, if $z_1 =$

$z_2 = i$, then $\operatorname{Log}(z_1 z_2)$ is not even defined. If $z_1 = z_2 = e^{3\pi i/4}$, then

$$z_1 z_2 = e^{3\pi i/2} = -i = e^{-\pi i/2},$$

so $\operatorname{Log}(z_1 z_2) = -\pi i/2 \neq \operatorname{Log}(z_1) + \operatorname{Log}(z_2) = 3\pi i/2$. However, on the slit plane we do have $\overline{\operatorname{Log}(z)} = \operatorname{Log}(\overline{z})$, $\operatorname{Log}(1/z) = -\operatorname{Log}(z)$, and $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$ if $z_1$, $z_2$ have positive real part (but perhaps $\operatorname{Log}(z_1 z_2 z_3) \neq \sum \operatorname{Log}(z_i)$ even if all $z_i$ have positive real part).

**Definition 2.15.** Let $\Omega$ be an open set in the complex plane and let $f$ be an analytic function on $\Omega$. A *logarithm* of $f$ is an analytic function $g$ on $\Omega$ whose exponential is $f$, *i.e.*, $g$ satisfies $e^{g(z)} = f(z)$ for all $z \in \Omega$.

We could write $g = \log f$, but due to the multi-valued nature of logarithms such notation should be used with care. (The ambiguity is in the imaginary part, so the notation $\operatorname{Re} \log f$ is well-defined, being just $\log |f|$.)

**Example 2.16.** The principal value logarithm $\operatorname{Log} z$ defined in (2.2) is a logarithm of the identity function $z$ on the slit plane.

**Example 2.17.** For an angle $\alpha$, omit the ray $\{re^{i\alpha} : r \geqslant 0\}$ from the plane. Write each number not on this ray uniquely in the form $re^{i\theta}$ where $\alpha < \theta < \alpha + 2\pi$, and define

$$\log z := \log r + i\theta,$$

where $\log r$ is the usual real logarithm. This logarithm differs from the principal value logarithm by an integral multiple of $2\pi$ on their common domain of definition. We will call any logarithm of this type, on a plane slit by a ray from the origin, a "slit logarithm."

**Example 2.18.** If $g$ is a logarithm of $f$, so is $g + 2\pi ik$ for each $k \in \mathbf{Z}$. There is never a unique logarithm of an analytic function.

**Example 2.19.** Let $g_1$ and $g_2$ be logarithms of the analytic functions $f_1$ and $f_2$. Then $g_1 + g_2$ is a logarithm of $f_1 f_2$. Indeed, $g_1 + g_2$ is analytic, and $e^{g_1(z)+g_2(z)} = e^{g_1(z)} e^{g_2(z)} = f_1(z) f_2(z)$.

**Theorem 2.20.** *Let $f \colon \Omega \to \mathbf{C}$ be analytic on the open set $\Omega$. If $g$ is a logarithm of $f$, then $g' = f'/f$.*

*Proof.* Informally, we can write $g = \log f$ and apply the chain rule.

More rigorously, differentiate the equation $f = e^g$ to get $f' = e^g g' = f g'$, so $g' = f'/f$. ∎

The expression $f'/f$ is called the logarithmic derivative of $f$, but note that one does *not* need $f$ to have a logarithm in order to construct this ratio. Any meromorphic function has a logarithmic derivative. Writing the product rule as

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

shows that the logarithmic derivative of a product is the sum of the logarithmic derivatives.

The construction of logarithmic derivatives will be useful to us, because of the following calculation:

$$f(s) = c_m(s-a)^m + c_{m+1}(s-a)^{m+1} + \cdots \Longrightarrow \frac{f'(s)}{f(s)} = \frac{m}{s-a} + \ldots, \qquad (2.3)$$

where $c_m \neq 0$, so the logarithmic derivative of $f$ is holomorphic except at zeros and poles of $f$, where it has a *simple* pole with residue equal to the order of vanishing of $f$ at $a$:

$$\mathrm{Res}_{s=a} \frac{f'(s)}{f(s)} = \mathrm{ord}_{s=a} f(s).$$

This suggests an analytic way of proving $f$ is holomorphic at $a$ and $f(a) \neq 0$: show $\mathrm{Res}_{s=a}(f'/f) = 0$. A meromorphic function on an open set is both holomorphic and nonvanishing if and only if its logarithmic derivative is holomorphic. In particular, a holomorphic function on an open set is nonvanishing if and only if its logarithmic derivative is holomorphic.

Because the multiplicity of a zero or pole is a residue of the logarithmic derivative, we can count zeros and poles inside a region by integration.

**Theorem 2.21 (Argument Principle).** *Let $f$ be meromorphic on a simple loop $\gamma$ and its interior, and be holomorphic and nonvanishing on $\gamma$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(s)}{f(s)}\, ds = N - P,$$

*where $N$ is the number of zeros of $f(s)$ inside of $\gamma$ and $P$ is the number of poles. Each zero and pole is counted with its multiplicity.*

In practice, $\gamma$ will be a circle or rectangle, so we don't have to deal with subtle definitions of what the "inside" of $\gamma$ means.

*Proof.* Apply the residue theorem. ∎

Since the integral is a real number, we only have to compute the imaginary part of the integral:

$$\frac{1}{2\pi}\,\mathrm{Im}\int_\gamma \frac{f'(s)}{f(s)}\,ds = N - P,$$

**Theorem 2.22.** *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on an open set $\Omega \subset \mathbf{C}$ that is simply connected. If $f$ vanishes nowhere on $\Omega$, then it has a logarithm.*

*Proof.* Intuitively, whatever "$\log f$" may turn out to be, its derivative ought to be

$$\frac{f'(z)}{f(z)}.$$

Since $f$ is nonvanishing, this ratio always makes sense. Since we know what the derivative of the putative logarithm of $f$ ought to be, we should obtain the logarithm itself by integrating $f'/f$. That's the main point, and the proof amounts to a technical verification that this intuitive idea works.

By passing to a connected component of $\Omega$, we can assume $\Omega$ is connected and open.

Pick $z_0 \in \Omega$. For $z \in \Omega$, let $\gamma_z$ be a continuous piecewise differentiable path from $z_0$ to $z$. (Connected open sets in the plane are path-connected, so there is such a path. In practice, when $\Omega$ is a disc or half-plane, this is geometrically obvious.) Choose $w_0 \in \mathbf{C}$ such that $e^{w_0} = f(z_0)$. We set

$$L_f(\gamma_z, z) := w_0 + \int_{\gamma_z} \frac{f'(w)}{f(w)}\,dw.$$

This will turn out to be a logarithm for $f$.

Since $\Omega$ is simply connected, this definition does not depend on the path $\gamma_z$ selected from $z_0$ to $z$, so we may write the function as $L_f(z)$. (It *does* depend on $z_0$, but that will remain fixed throughout the proof.) Note $L_f(z_0) = w_0$.

To show $L_f'(z) = f'(z)/f(z)$, we have for small $h$

$$\begin{aligned} L_f(z+h) - L_f(z) &= \int_{[z,z+h]} R(z)\,dw + \int_{[z,z+h]} (R(w) - R(z))\,dw \\ &= R(z)h + \int_{[z,z+h]} (R(w) - R(z))\,dw, \end{aligned}$$

where $R(z) = f'(z)/f(z)$ and $[z, z+h]$ denotes the straightline path between $z$ and $z+h$. Now divide by $h$ and let $h \to 0$.

To show $L_f$ is a logarithm of $f$, *i.e.*, $e^{L_f(z)} = f(z)$, we show the ratio is constant:

$$\frac{d}{dz}e^{-L_f(z)}f(z) = e^{-L_f(z)}f'(z) - f(z)L_f'(z)e^{-L_f(z)} = e^{-L_f(z)}f'(z) - f'(z)e^{-L_f(z)},$$

which is 0. We're on a connected set, so it follows that $f(z) = ce^{L_f(z)}$ for some constant $c$. We want to show the constant $c$ is 1. It suffices to check the equation at one point in $\Omega$. At $z_0$ we have $f(z_0) = ce^{L_f(z_0)}$. Is $c = 1$? By our construction, $L_f(z_0) = w_0$ and $e^{w_0} = f(z_0)$, so $c = 1$. ■

**Example 2.23.** Let $D$ be an open disc in the plane that does not contain 0. The function $z$ is nonvanishing on $D$, so there is a logarithm function $\ell(z)$ on $D$, *i.e.*, $\ell$ is analytic on $D$ and $e^{\ell(z)} = z$. A construction of a logarithm is $\ell(z) = w_0 + \int_{r_z} dw/w$, where $r_z$ is the radial path from the center of $D$ to $z$ and $w_0$ is a constant chosen so $e^{w_0}$ is the center of $D$.

**Example 2.24.** Let $\Delta_+ = \{a + bi : a^2 + b^2 < 1, b > 0\}$ be the upper part of the unit disc and $f(z) = z^8$, which is nonvanishing on $\Delta_+$. Since $f'(z)/f(z) = 8/z$, a logarithm of $f$ on $\Delta_+$ is $L_f(z) = w_0 + 8\int_{\gamma_z} dw/w$ where $\gamma_z$ is a path in $\Delta_+$ from (say) $i/2$ to $z$ and $w_0$ is chosen so $e^{w_0} = i/2$, *e.g.*, $w_0 = \log(1/2) + i\pi/2$.

Notice that although $\Delta_+$ is simply connected, the image of $\Delta_+$ under $f$ is the punctured unit disc $\{w : 0 < |w| < 1\}$, which has a hole. The principal value logarithm $\operatorname{Log} w$ is not defined on the whole punctured unit disc, so it is *false* that $L_f(z)$ equals the composite $\operatorname{Log}(f(z))$, as the latter is not always defined. It is true by our choice of $w_0$ that $L_f(z) = 8 \operatorname{Log} z$.

The lesson from Example 2.24 is that a logarithm of the function $f$ is not typically a composite of the slit logarithm and the function $f$ on the whole domain. If we are only concerned with an analytic function locally, then the situation is simpler:

**Theorem 2.25.** *Let $f \colon \Omega \to \mathbf{C}$ be an analytic function on the open set $\Omega$. If $f(z_0) \neq 0$, then $f$ has a local logarithm near $z_0$. That is, there is an analytic function $g$ on some neighborhood of $z_0$ in $\Omega$ such that $e^g = f$ on this neighborhood.*

*Proof.* Since $f(z_0) \neq 0$, select a small disc $D$ around $f(z_0)$ that doesn't contain the origin. There is a corresponding small disc $D_0 \subset \Omega$ around $z_0$ such that $f(D_0) \subset D$. There is a logarithm function defined on $D$, since $D$ lies in some plane slit by a ray from the origin and we can easily write down logarithms on such domains. The composite of such a logarithm with $f$ is clearly a logarithm of $f$, *i.e.*, it is analytic and its exponential is $f$. ■

This proof is simple because we can just compose $f$ and some slit logarithm function. To globalize the result from small discs to general $\Omega$, as in Theorem 2.22, is technical precisely because the ordinary logarithm function is not a well-defined analytic function on $\mathbf{C}^\times$. Still, in almost all situations where we may want to apply Theorem 2.22, Theorem 2.25 will suffice.

**Theorem 2.26.** *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on a connected open set. Any two logarithms of $f$ differ by an integral multiple of $2\pi i$.*

The theorem is vacuous if $f$ doesn't admit a logarithm.

*Proof.* The difference of two logarithms of $f$ is a function whose exponential is identically 1, so the difference takes values in the discrete set $2\pi i\mathbf{Z}$. By continuity of logarithms and connectedness of $\Omega$, the difference is constant. ■

In practice, we will only be discussing logarithms of analytic functions defined on regions, *i.e.*, connected open sets.

By Theorem 2.26, a logarithm of $f(z)$ is determined by its value at one point, or by its limit as $z$ tends to $\infty$ in some fixed direction, if the function settles down to a constant value in that direction.

A consequence of the previous few theorems is that every continuous logarithm of an analytic function is analytic.

**Corollary 2.27.** *Let $f\colon \Omega \to \mathbf{C}$ be an analytic function on a simply connected region (such as a disc or half-plane). Every continuous logarithm of $f$ is an analytic logarithm. That is, every continuous function $\ell_f$ on $\Omega$ such that $e^{\ell_f(z)} = f(z)$ on $\Omega$ is analytic.*

*Proof.* Since $f(z) = e^{\ell_f(z)}$, $f$ is nonvanishing, so $f$ admits an analytic logarithm, say $L_f$. (Recall that this logarithm was constructed in Theorem 2.22, via path integrals of $f'/f$. Or, since analyticity is a local property, we may suppose $\Omega$ is a suitably small disc and appeal to the simpler Theorem 2.25.) The proof of Theorem 2.26 only used continuity of the two logarithms, not their analyticity. So that proof shows $\ell_f - L_f$ is a constant. Since $L_f$ is analytic, $\ell_f$ must be analytic. ■

Square roots should be expressible as $\sqrt{z} = e^{(1/2)\log z}$, so we can construct square roots of analytic functions if we can construct analytic logarithms.

**Definition 2.28.** Let $f\colon \Omega \to \mathbf{C}$ be analytic. A *square root* of $f$ is an analytic function $g$ on $\Omega$ such that $g^2 = f$.

For example, $e^{z/2}$ is a square root of $e^z$ and $z$ is a square root of $z^2$. Unlike logarithms of analytic functions, square roots of analytic functions can vanish. (The constant function 0 is a square root of itself; in no other case will a square root vanish except at isolated points.) In practice we will be considering only nonvanishing functions (at least in the region where we may construct logarithms or square roots), so part i) of the following theorem covers the cases that will matter.

**Theorem 2.29.** *i) If $\ell_f$ is a logarithm of the analytic function $f$, then $g(z) = e^{(1/2)\ell_f(z)}$ is a square root of $f$. In particular, every nonvanishing analytic function on a simply connected region has a square root.*

*ii) If $g_1$ and $g_2$ are square roots of $f$, then $g_1 = g_2$ or $g_1 = -g_2$.*

*iii) If $f\colon \Omega \to \mathbf{C}$ is an analytic function on an open set, every continuous square root is an analytic square root.*

*Proof.* Exercise 2.3.5. ■

**Example 2.30.** On the upper half-plane $\mathfrak{H} = \{a + bi : b > 0\}$, with variable element $\tau$, the function $f(\tau) = \tau/i$ is nonvanishing, so it has an analytic square root, in fact exactly two of them. Writing $\tau = re^{i\theta}$ where $0 < \theta < \pi$, we have $\tau/i = re^{i(\theta-\pi/2)}$. Set $\sqrt{\tau/i} = \sqrt{r}e^{i(\theta/2-\pi/4)}$. This is certainly a continuous square root function. It must be analytic by part iii) of Theorem 2.29. Notice this particular analytic square root is positive on the imaginary axis, namely if $\tau = bi$ for $b > 0$ then $\sqrt{\tau/i}$ is the real positive square root of $b$.

The next complex analytic topic is the Gamma function. This is an important function in both pure and applied mathematics. In number theory it arises in the functional equations of $\zeta(s)$ and $L(s,\chi)$. It is not naturally defined by a power series expansion, but by an integral formula.

We begin with the equation

$$n! = \int_0^\infty x^n e^{-x}\, dx. \tag{2.4}$$

This is clear when $n = 0$, since $\int_0^\infty e^{-x}\, dx = 1$. Integrating by parts ($u = x^n$, $dv = e^{-x}dx$) gives (2.4) for larger $n$ by induction. (Euler discovered this formula, but wrote it as $n! = \int_0^1 \log(1/y)^n\, dy$.) The right side of (2.4) makes sense for nonintegral values of $n$.

**Definition 2.31.** For complex numbers $s$ with $\mathrm{Re}(s) > 0$, set

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}\,dx = \int_0^\infty x^s e^{-x}\,\frac{dx}{x}.$$

We'll check below that this integral does converge. In addition to checking the behavior of the integrand $x^{s-1}e^{-x}$ near $x = \infty$, when $0 < \mathrm{Re}(s) < 1$ we have to pay attention to behavior near $x = 0$. This definition does not make sense if $\mathrm{Re}(s) \leqslant 0$. But we will see that $\Gamma(s)$ can be continued to a meromorphic function on $\mathbf{C}$.

For a positive integer $n$, $\Gamma(n) = (n-1)!$. It may seem more reasonable to work with $\Pi(s) := \int_0^\infty x^s e^{-x}\,dx$, since $\Pi(n) = n!$, and this is the factorial generalization used by Riemann in his paper on the zeta-function. The function $\Gamma(s) = \Pi(s-1)$ was introduced by Legendre.

The Gamma function is an example of an integral of the form

$$\int_0^\infty f(x)\,\frac{dx}{x}. \tag{2.5}$$

Such integrals are invariant under multiplicative translations: for every $c > 0$,

$$\int_0^\infty f(cx)\,\frac{dx}{x} = \int_0^\infty f(x)\,\frac{dx}{x}.$$

Similarly, for $c > 0$

$$\int_{-\infty}^\infty f(cx)\,\frac{dx}{|x|} = \int_{-\infty}^\infty f(x)\,\frac{dx}{|x|}.$$

(The additive analogue is the more familiar $\int_{-\infty}^\infty g(x+c)\,dx = \int_{-\infty}^\infty g(x)\,dx$.)

To check the integral defining $\Gamma(s)$ makes sense, we take absolute values and reduce to the case of *real* $s > 0$. Behavior at 0 and $\infty$ are isolated by splitting up the integral into two pieces, from 0 to 1 and from 1 to $\infty$:

$$\int_0^1 x^{s-1}e^{-x}\,dx + \int_1^\infty x^{s-1}e^{-x}\,dx.$$

For $x > 0$, $x^{s-1}e^{-x} \leqslant x^{s-1}$ and $x^{s-1}$ is integrable on $[0,1]$ if $s > 0$, so the first integral converges. For the second integral, remember that exponentials grow much faster than powers. We write $e^{-x} = e^{-x/2}e^{-x/2}$ and then have

$$x^{s-1}e^{-x} \leqslant C_s e^{-x/2}$$

on $[1,\infty)$ for some constant $C_s$, so the second integral converges. Actually, the second integral converges for *every* $s \in \mathbf{C}$, not just when $\mathrm{Re}(s) > 0$.

One important special value of $\Gamma$ at a noninteger is at $s = 1/2$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}\,dx = 2\int_0^\infty e^{-u^2}\,du = \int_{-\infty}^\infty e^{-u^2}\,du.$$

A common method of calculating $\int_{-\infty}^\infty e^{-u^2}\,du$ is by squaring and then passing to polar coordinates. For $I = \int_{-\infty}^\infty e^{-u^2}\,du$,

$$\begin{aligned} I^2 &= \int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\,dy \\ &= \int_0^\infty\int_0^{2\pi} e^{-r^2}r\,dr\,d\theta \\ &= \int_0^\infty re^{-r^2}\,dr\cdot\int_0^{2\pi}d\theta \\ &= \frac{1}{2}\cdot 2\pi \\ &= \pi, \end{aligned}$$

so (since $I > 0$) $I = \sqrt{\pi}$. Therefore

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

By Theorem 2.10, $\Gamma(s)$ is analytic on $\mathrm{Re}(s) > 0$. Now we extend $\Gamma$ meromorphically to the entire complex plane.

For complex $s$ with positive real part, an integration by parts ($u = x^s, dv = e^{-x}dx$) yields the *functional equation*

$$\Gamma(s+1) = s\Gamma(s). \tag{2.6}$$

This generalizes $n! = n(n-1)!$.

**Theorem 2.32.** *The function $\Gamma$ extends to a meromorphic function on $\mathbf{C}$ whose only poles are at the integers $0, -1, -2, \ldots,$ where the poles are simple with residue at $-k$ equal to $(-1)^k/k!$.*

*Proof.* Use (2.6) to extend $\Gamma$ step-by-step to a meromorphic function on $\mathbf{C}$. The residue calculation follows from the functional equation and induction. ■

There are a few important relations that $\Gamma$ satisfies besides (2.6). For example, since $\Gamma(s)$ has simple poles at $0, -1, -2, \ldots$, $\Gamma(1-s)$ has simple poles at $1, 2, 3, \ldots$. The product $\Gamma(s)\Gamma(1-s)$ therefore has simple poles precisely at the integers, just like $1/\sin(\pi s)$. What is the relation between these functions? Also, $\Gamma(s/2)$ has simple poles at $0, -2, -4, -6, \ldots$ and $\Gamma((s+1)/2)$ has simple poles at $-1, -3, -5, \ldots$. Therefore $\Gamma(s/2)\Gamma((s+1)/2)$ has simple poles at $0, -1, -2, \ldots$, just like $\Gamma(s)$. What is the relation between $\Gamma(s)$ and the product $\Gamma(s/2)\Gamma((s+1)/2)$? In general, knowing where a functions has its poles (and zeros) is not enough to characterize it. One can always introduce an arbitrary multiplicative constant. That is the only purely algebraic modification, but we can also introduce a factor of an arbitrary nonvanishing entire function, say $e^{g(z)}$ where $g(z)$ is entire. For $g(z)$ linear, this factor looks like $ae^{bz}$. This is the type of factor we'll need to introduce to relate the above Gamma functions with each other.

To work with factorization questions, it is more convenient to use a different formula for $\Gamma(s)$ than the integral definition. The integral is an additive formula for $\Gamma(s)$. We now turn to two multiplicative formulas, valid in the whole complex plane. They are usually attributed to Gauss and Weierstrass, respectively, although neither one was the first to discover these formulas.

**Lemma 2.33.** *For every $s \in \mathbf{C}$,*

$$\Gamma(s) = \lim_{n\to\infty} \frac{n! n^s}{s(s+1)\cdots(s+n)} = \frac{e^{\gamma s}}{s} \prod_{n \geqslant 1} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n},$$

*where $\gamma = \lim_{n\to\infty} 1 + 1/2 + \cdots + 1/n - \log n \approx .5772156649015\ldots$ is Euler's constant.*

*Proof.* See [**?**, Chap. XV] or [**?**, Chap. 2]. For $s = 0, \pm 1, \pm 2, \ldots$ the products are set equal to $\infty$. ∎

The Weierstrass product can also be written as

$$\Gamma(s+1) = e^{\gamma s} \prod_{n \geqslant 1} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}.$$

The exponent $\gamma$ in the Weierstrass product has the characterizing role of guaranteeing that $\Gamma(s+1)/\Gamma(s)$ equals $s$ rather than some other constant multiple of $s$.

**Theorem 2.34.** *For all complex numbers $s$,*

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)},\qquad \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)=\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s).$$

The first equation is called the reflection formula, and the second is the duplication formula.

*Proof.* To get the first identity, write it as $\Gamma(s)\Gamma(-s)=-\pi/(s\sin(\pi s))$. Compute the left side using the product in Lemma 2.33 and the right side using the product for the sine function,

$$\sin s = s\prod_{n\geqslant 1}\left(1-\frac{s^2}{n^2\pi^2}\right), \tag{2.7}$$

replacing $s$ with $\pi s$.

For the second identity, use Gauss' limit formula in Lemma 2.33 to compute $\Gamma(s/2)\Gamma((s+1)/2)$. The result is $c\Gamma(s)/2^s$, and setting $s=1$ shows $c=2\sqrt{\pi}$. ■

**Corollary 2.35.** *The function $\Gamma(s)$ has no zeros.*

*Proof.* The poles of $\Gamma(s)$ are at the integers $\leqslant 0$. Since $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$, we see that $\Gamma(s)\neq 0$ for $s\notin \mathbf{Z}$. We already know the values of $\Gamma(s)$ at the integers, where it is not zero. ■

In particular, while $\Gamma(s)$ is meromorphic, $1/\Gamma(s)$ is entire. So if $f(s)\Gamma(s)$ is an entire function, so is $f(s)$.

We conclude with a basic asymptotic formula for the Gamma function.

**Theorem 2.36 (Stirling's Formula).** *Fix positive $\varepsilon<\pi$. As $|s|\to\infty$ in a sector $\{s : |\mathrm{Arg}(s)|\leqslant \pi-\varepsilon\}$,*

$$\Gamma(s)=\sqrt{2\pi}s^{s-1/2}e^{-s}e^{\mu(s)}=\sqrt{2\pi}e^{(s-1/2)\,\mathrm{Log}\,s-s+\mu(s)},$$

*where the error term in the exponent satisfies $\mu(s)=O(1/|s|)$ as $|s|\to\infty$, the constant in the $O$-symbol depending on the sector. In particular, $\Gamma(s)\sim\sqrt{2\pi}s^{s-1/2}e^{-s}$ as $s\to\infty$ in such a sector.*

*Proof.* See [?, Chap. XV] or [?, Chap. 2]. The proof involves producing an exact error estimate for $\log\Gamma(s)$, and leads to the additional formula

$$\frac{\Gamma'(s)}{\Gamma(s)}=\mathrm{Log}\,s-\frac{1}{2s}+O(1/|s|^2)=\mathrm{Log}\,s+O(1/|s|)$$

by differentiation of the exact error estimate. ■

An important application of the complex Stirling's formula is to the growth of the Gamma function on vertical lines. (We call this vertical growth.) For $\sigma > 0$, the integral formula for the Gamma function shows $|\Gamma(\sigma + it)| \leqslant |\Gamma(\sigma)|$ for any real $t$, so vertical growth is bounded. But in fact it is exponentially decaying, as follows.

**Corollary 2.37.** *For fixed $\sigma$, $|\Gamma(\sigma + it)| \sim \sqrt{2\pi}e^{-(\pi/2)|t|}|t|^{\sigma-1/2}$ as $|t| \to \infty$. More generally, this estimate applies in any vertical strip of the complex plane, and is uniform with respect to $\sigma$ in that strip.*

*Proof.* By iterating the functional equation $\Gamma(s + 1) = s\Gamma(s)$, we can reduce to the case of a closed vertical strip in the half-plane $\mathrm{Re}(s) > 0$. We leave this reduction step to the reader, and for simplicity we only treat the case of a single vertical line rather than a strip.

Since $|\Gamma(\sigma + it)| = |\Gamma(\sigma - it)|$, we only need to consider $t > 0$, $t \to \infty$.

For $\sigma > 0$ fixed and $t \to \infty$, Stirling's formula (in the form of Theorem 2.36) gives

$$\begin{aligned}
|\Gamma(\sigma + it)| &\sim \sqrt{2\pi}e^{\mathrm{Re}((\sigma-1/2+it)\,\mathrm{Log}(\sigma+it))}e^{-\sigma} \\
&= \sqrt{2\pi}e^{(\sigma-1/2)(1/2)\log(\sigma^2+t^2)-t\arctan(t/\sigma)-\sigma} \\
&\sim \sqrt{2\pi}e^{(\sigma-1/2)\log t-(\pi/2)t}
\end{aligned}$$

since $t(\pi/2 - \arctan(t/\sigma)) \to \sigma$ as $t \to \infty$. ■

As a particular example, note that for real $a, b$, and $c$ with $a \neq 0$, the exponential factors in the Stirling estimates for $|\Gamma(as + b)|$ and $|\Gamma(-as + c)|$ (when $t = \mathrm{Im}(s) \to \infty$) are identical, so

$$\frac{\Gamma(as + b)}{\Gamma(-as + c)} = O(t^M) \tag{2.8}$$

as $t \to \infty$, where $M$ depends on the parameters $a, b$, and $c$. This polynomial upper bound on growth is convenient when estimating the growth of $\zeta(s)$ and $L(s, \chi)$ in vertical strips.

Exercises for Section 2.3

1. Let $g\colon \Omega \to \mathbf{C}$ be holomorphic on an open set $\Omega$, with $a \in \Omega$. If $g$ vanishes to order $m$ at $a$ (so $g(z) = c_m(z - a)^m + c_{m+1}(z - a)^{m+1} + \ldots, c_m \neq 0$),

show that near $a$, $g$ is the $m$th power of a holomorphic function that has a simple zero at $a$.

2. a) Prove Theorem 2.11 directly from the definition of the complex derivative, avoiding local series expansions.

   b) If $f$ is analytic and on an open interval of the real line it is real-valued, show the maximal domain of analyticity of $f$ is symmetric about the real axis, and $\overline{f(s)} = f(\overline{s})$ for all $s$ in the domain of $f$. In particular, $|f(\overline{s})| = |f(s)|$. (So the nonreal zeros of $f$ come in conjugate pairs.) This applies, for instance, to any function that is an analytic continuation of a power series with real coefficients. It also applies to the Gamma function.

3. Show $\Gamma(s) = 1/s - \gamma + \dots$ for $s$ near 0.

4. Fix $\sigma_0 \in \mathbf{R}$, and let

$$M(r, \sigma_0) = \sup_{\substack{|s|=r \\ \mathrm{Re}(s) \geqslant \sigma_0}} \log |\Gamma(s)|.$$

   Show $M(r, \sigma_0) \sim r \log r \sim \log \Gamma(r)$ as $r \to \infty$.

5. Prove Theorem 2.29.

6. Prove the reflection formula in Theorem 2.34 using Gauss' limit formula from Lemma 2.33 rather than the infinite product.

7. a) Show $\Gamma'(1) = -\gamma$ using either of the formulas in Lemma 2.33. (Hint: Logarithmic derivatives.)

   b) Show $\Gamma'(s)/\Gamma(s) = \lim_{n\to\infty} \log n - \sum_{k=0}^{n} 1/(k+s)$ for $s \neq 0, -1, -2, \dots$.

   c) Show

$$\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)' = \sum_{n \geqslant 0} \frac{1}{(n+s)^2},$$

   and compute $\Gamma''(1)$.

8. Show $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2\log 2)$.

## 2.4 Dirichlet Series

The examples of $\zeta(s)$ and $L(s,\chi)$ lead to the consideration of functions of the form

$$f(s) = \sum_{n\geqslant 1} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots,$$

where the $a_n$ are complex numbers and $s$ is a complex variable. Such functions are called *Dirichlet series*. We call $a_1$ the *constant term*.

A Dirichlet series will often be written as $\sum a_n n^{-s}$, with the index of summation understood to start at $n = 1$. Similarly, $\sum a_p p^{-s}$ runs over the primes, and $\sum a_{p^k} p^{-ks}$ runs over the prime powers *excluding* 1. (Not counting 1 as a prime power in that notation is reasonable in light of the way Dirichlet series that run over prime powers arise in practice, without a constant term.) A Dirichlet series over the prime powers that excludes the primes will be written $\sum_{p,k\geqslant 2} a_{p^k} p^{-ks}$.

The use of $s$ as the variable in a Dirichlet series goes back to Dirichlet, who took $s$ to be real and positive. Riemann emphasized the importance of letting $s$ be complex. The convention of using $\sigma$ and $t$ for the real and imaginary parts of $s$ seems to have become common at the beginning of the 20th century,[1] and was universally adopted through the influence of Landau's *Handbuch* [**?**] (1909).

**Example 2.38.** If $a_n = 1$ for all $n$, $f(s) = \zeta(s)$, which converges for $\sigma > 1$. It does not converge at $s = 1$.

**Example 2.39.** If $a_n = \chi(n)$ for a Dirichlet character $\chi$, $f(s)$ is the $L$-function $L(s,\chi)$ and converges absolutely for $\sigma > 1$. Note $L(s,\chi_4)$ converges for real $s > 0$ since the Dirichlet series is then an alternating series. A general Dirichlet character does not take alternating values $\pm 1$, but we'll see that as long as $\chi$ is not a trivial Dirichlet character, $L(s,\chi)$ converges (though not absolutely) when $0 < \mathrm{Re}(s) \leqslant 1$.

**Example 2.40.** For a Dirichlet series $\sum a_n n^{-s}$ we can consider $\sum \chi(n) a_n n^{-s}$ for some Dirichlet character $\chi$. We call the latter function a *twist* of the former, by $\chi$. So $L(s,\chi)$ is a twist of the zeta-function.

**Example 2.41.** If $a_n = n^k$ for an integer $k$, then $f(s) = \zeta(s-k)$ converges for $\sigma > k + 1$.

[1] This notation is not due to Riemann.

**Example 2.42.** If $a_n = 1/n^n$ then $f(s)$ converges for all $s$. Unlike power series, no Dirichlet series that arises naturally converges on the whole complex plane, so you should not regard examples of this sort as important.

**Theorem 2.43.** *If $\sum a_n n^{-s_0}$ converges absolutely, then $\sum a_n n^{-s}$ converges absolutely for $\sigma \geqslant \mathrm{Re}(s_0)$.*

*Proof.* Use the comparison test. ∎

Our first task is to show this theorem remains true when we weaken absolute convergence to convergence, except the inequality in the conclusion will become strict. The following estimate will be useful.

**Lemma 2.44.** *For $\sigma = \mathrm{Re}(s) > 0$ and $0 < a < b$,*

$$\left|\frac{1}{a^s} - \frac{1}{b^s}\right| \leqslant \frac{|s|}{\sigma}\left(\frac{1}{a^\sigma} - \frac{1}{b^\sigma}\right) \leqslant \frac{|s|(b-a)}{a^{\sigma+1}}.$$

*Proof.* We integrate

$$\int_a^b \frac{dx}{x^{s+1}} = \frac{1}{s}\left(\frac{1}{a^s} - \frac{1}{b^s}\right),$$

so

$$\left|\frac{1}{a^s} - \frac{1}{b^s}\right| \leqslant |s| \int_a^b \frac{dx}{x^{\sigma+1}} = \frac{|s|}{\sigma}\left(\frac{1}{a^\sigma} - \frac{1}{b^\sigma}\right).$$

The second upper bound in the theorem follows by either the Mean Value Theorem or an easy estimate on the integral. ∎

A useful form of this estimate is for consecutive integers:

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \leqslant \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right) \leqslant \frac{|s|}{n^{\sigma+1}}. \tag{2.9}$$

**Theorem 2.45 (Jensen–Cahen).** *Let $A_n = a_1 + a_2 + \cdots + a_n$. If $\{A_n\}$ is a bounded sequence, then $\sum a_n n^{-s}$ converges and is an analytic function on the half-plane $\sigma > 0$, with the derivative computable termwise. Convergence is absolute on the half-plane $\sigma > 1$.*

*More generally, if the series $\sum a_n n^{-s}$ converges at $s_0 = \sigma_0 + it_0$, or just has bounded partial sums at $s_0$, then $\sum a_n n^{-s}$ converges and is analytic for $\sigma > \sigma_0$, with the derivative computable termwise. Convergence is absolute on the half-plane $\sigma > \sigma_0 + 1$.*

Note the profound difference with convergence of power series. If a power series $\sum c_n(z-a)^n$ converges at the point $z_0$, then we know only that the series converges on the disc $|z-a| < |z_0-a|$, a bounded region. But if a Dirichlet series $\sum a_n n^{-s}$ converges at a point $s_0$, we obtain convergence on the half-plane $\sigma > \mathrm{Re}(s_0)$, which is an unbounded region. This points to the inherently nonlocal nature of a Dirichlet series, unlike the tool of local power series expansions that pervades complex function theory. For this reason Dirichlet series are not a general purpose tool in complex analysis.

Unless otherwise specified, the undecorated term "half-plane" always refers to a right half-plane $\sigma > \sigma_0$ or $\sigma \geqslant \sigma_0$.

*Proof.* In the general case, we set $b_n = a_n n^{-s_0}$, so the partial sums $b_1 + \cdots + b_n$ are bounded and we are in the special case $s_0 = 0$. Then the conclusions about the series $\sum b_n n^{-s} = \sum a_n n^{-(s+s_0)}$ give the conclusions about $\sum a_n n^{-s}$. We are reduced to the case $s_0 = 0$, so we may assume $a_1 + \cdots + a_n$ is bounded. We want convergence for complex $s$ with $\mathrm{Re}(s) > 0$.

A problem with using the series $\sum a_n n^{-s}$ directly is that its convergence is not always absolute, *e.g.*, if $a_n = (-1)^n$ and $s = 1$. It is easier to work with a series that is absolutely convergent instead.

Since $a_n = A_n - A_{n-1}$ for $n \geqslant 2$, by partial summation

$$\begin{aligned} \sum_{n=1}^{N} \frac{a_n}{n^s} &= a_1 + \sum_{n=2}^{N} \frac{A_n - A_{n-1}}{n^s} \\ &= \frac{A_N}{(N+1)^s} + \sum_{n=1}^{N} A_n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right). \end{aligned}$$

Since $\{A_n\}$ is bounded, $A_N/(N+1)^s \to 0$ as $N \to \infty$ when $\sigma = \mathrm{Re}(s) > 0$.

So for $\sigma > 0$, $\sum a_n n^{-s}$ converges if and only if $\sum A_n(n^{-s} - (n+1)^{-s})$ converges, in which case they are equal. Moreover,

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^s} - \sum_{n=1}^{N} A_n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right| = \left| \frac{A_N}{(N+1)^s} \right| \leqslant \frac{C}{(N+1)^\sigma}, \tag{2.10}$$

where we take $|A_n| \leqslant C$ for all $n$.

Let

$$f_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}, \quad g_N(s) = \sum_{n=1}^{N} A_n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right).$$

If $s$ runs over a compact subset $K$ of $\{s : \sigma > 0\}$ then for some $b > 0$ we have $\sigma \geqslant b$ for all $s \in K$, so the difference $|f_N(s) - g_N(s)|$ in (2.10) is at most $C/(N+1)^b$ and therefore tends to 0 uniformly (as $N \to \infty$) on each compact subset of $\{s : \sigma > 0\}$.

The sum involving $A_n$ rather than the $a_n$ turns out to be easier to estimate, using Lemma 2.44:

$$\begin{aligned}\sum_{n\geqslant 1}\left|A_n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\right| &\leqslant \sum_{n\geqslant 1}\frac{|A_n||s|}{n^{\sigma+1}} \\ &\leqslant C|s|\sum_{n\geqslant 1}\frac{1}{n^{\sigma+1}}.\end{aligned}$$

So for $\mathrm{Re}(s) > 0$, the series $g(s) := \sum A_n(n^{-s} - (n+1)^{-s})$ converges absolutely, hence $\sum a_n n^{-s}$ converges for $\mathrm{Re}(s) > 0$ (but not necessarily absolutely). We might call $g(s)$ the absolutely convergent formula for $f(s)$.

To show $\sum a_n n^{-s}$ is analytic for $\sigma > 0$, and to differentiate it termwise, we will show it is the *uniform* limit of $f_N(s)$ on every compact subset $K$ of $\sigma > 0$. (Analyticity then follows from Theorem 2.9.) Actually, we will show $\sum a_n n^{-s}$ is the uniform limit of $g_N(s)$. This is equivalent, since $f_N(s) - g_N(s) \to 0$ uniformly on $K$.

For $\sigma > 0$, let

$$f(s) = \sum_{n\geqslant 1}\frac{a_n}{n^s} = \sum_{n\geqslant 1}A_n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right). \tag{2.11}$$

Then by Lemma 2.44,

$$\begin{aligned}\left|f(s)-\sum_{n=1}^{N}A_n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\right| &= \left|\sum_{n\geqslant N+1}A_n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\right| \\ &\leqslant C|s|\sum_{n\geqslant N+1}\frac{1}{n^{\sigma+1}}.\end{aligned}$$

Choose a compact subset $K$ of the half-plane $\sigma > 0$. As before, for all $s \in K$, $\sigma \geqslant b > 0$ for some number $b$. Also, as $K$ is bounded, $|s| \leqslant M$ for some $M$.

Thus

$$\left|f(s)-\sum_{n=1}^{N}A_n\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\right| \leqslant CM\sum_{n\geqslant N+1}\frac{1}{n^{b+1}},$$

which $\to 0$ as $N \to \infty$, uniformly in $s \in K$.

A uniform limit of analytic functions on compact subsets can be differentiated termwise, so for $m \geqslant 1$

$$\frac{d^m}{ds^m}\left(\sum_{n\geqslant 1}\frac{a_n}{n^s}\right) = \sum_{n\geqslant 2}\frac{a_n(-\log n)^m}{n^s} \quad \text{on } \sigma > 0.$$

Since $A_n$ is a bounded sequence, the numbers $a_n = A_n - A_{n-1}$ are bounded, so the series $\sum |a_n n^{-(1+\varepsilon)}|$ converges for all $\varepsilon > 0$. Therefore $\sum a_n n^{-s}$ converges absolutely if $\sigma > 1$. ■

**Example 2.46.** The zeta-function is analytic for $\sigma > 1$, with

$$\zeta'(s) = -\sum_{n\geqslant 2}\frac{\log n}{n^s}, \quad \zeta''(s) = \sum_{n\geqslant 2}\frac{(\log n)^2}{n^s}.$$

Although

$$L\left(\frac{1}{2},\chi_4\right) = \sum_{n\geqslant 1}\frac{\chi_4(n)}{\sqrt{n}}, \qquad L'\left(\frac{1}{3},\chi_4\right) = -\sum_{n\geqslant 2}\frac{\chi_4(n)\log n}{\sqrt[3]{n}},$$

these sums are *not* absolutely convergent, and this marks a striking difference with power series. A power series is absolutely convergent on the interior of its disc of convergence, but a Dirichlet series can converge nonabsolutely on a vertical strip. If this seems strange when compared to the behavior of power series, then keep in mind that analyticity is linked not to absolute convergence, but to uniform convergence on compact sets, and the latter property is shared by both power series and Dirichlet series in the open sets where each converges.

**Theorem 2.47.** *Let $\chi\colon (\mathbf{Z}/(m))^\times \to \mathbf{C}$ be a nontrivial Dirichlet character, so $\chi(a) \neq 1$ for some unit $a \bmod m$. Set $\chi(n) = 0$ if $(n,m) > 1$. The Dirichlet series*

$$L(s,\chi) = \sum_{n\geqslant 1}\frac{\chi(n)}{n^s}$$

*converges for $\sigma > 0$. It converges absolutely for $\sigma > 1$.*

*Proof.* Absolute convergence on $\sigma > 1$ is clear. To get convergence for $\sigma > 0$, we show the partial sums $\chi(1) + \chi(2) + \cdots + \chi(n)$ are all bounded, so Theorem 2.45 applies.

Since $\chi$ is periodic, it suffices to show the sum over a full period vanishes:

$$\sum_{k=N}^{N+m-1} \chi(k) = \sum_{k \in (\mathbf{Z}/(m))^\times} \chi(k) = 0.$$

Let $S$ be this sum. Choosing a unit $a$ mod $m$ so that $\chi(a) \neq 1$,

$$\chi(a)S = \sum_{k \in (\mathbf{Z}/(m))^\times} \chi(ak) = \sum_{k \in (\mathbf{Z}/(m))^\times} \chi(k) = S,$$

so $S = 0$. ■

For the rest of this section, we put the background material on infinite products and logarithms of analytic functions to work on $\zeta(s)$ and $L(s,\chi)$.

**Theorem 2.48.** *For* $\mathrm{Re}(s) > 1$*,*

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \exp\left(\sum_{p,k} \frac{1}{kp^{ks}}\right) \neq 0$$

*and*

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}} = \exp\left(\sum_{p,k} \frac{\chi(p^k)}{kp^{ks}}\right) \neq 0.$$

*Proof.* To justify expanding

$$\prod_p \frac{1}{1-p^{-s}} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$$

into a Dirichlet series, which will turn out to be the zeta-function, we apply Theorem 2.6 with $z_n = p_n^{-s}$ as $p_n$ runs through the primes. The hypothesis $\sum |z_n| < \infty$ for Theorem 2.6, in this application, forces $\mathrm{Re}(s) > 1$.

The equality between the product and the exponential occurs within the proof of Theorem 2.6, and the case of $L(s,\chi)$ is left as an exercise. ■

The expression of the zeta-function as an exponential or Euler product as in Theorem 2.48 should be regarded as a basic structural ingredient, in some sense more fundamental than the usual Dirichlet series definition. Of course the Dirichlet series is important, *e.g.*, it will be used in proving the analytic continuation of $\zeta(s)$. However, some similar types of functions that are significant for number theory, such as zeta-functions of varieties over finite fields or Artin

$L$-functions, can only be defined as exponentials or Euler products, not as a series. While a series expression in these two cases is possible, it is not a good starting point.

Using the Euler product and the fact that $(fg)'/(fg) = f'/f + g'/g$, with passage to the limit, we get from the Euler product of $\zeta(s)$ a Dirichlet series for $\zeta'(s)/\zeta(s)$ and $L'(s,\chi)/L(s,\chi)$:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{p^k} \frac{\log p}{p^{ks}}, \qquad \frac{L'(s,\chi)}{L(s,\chi)} = -\sum_{p^k} \frac{\chi(p^k)\log p}{p^{ks}}. \tag{2.12}$$

Exercises for Section 2.4

1. If $\sum_{n \leqslant x} a_n = O(x^\delta)$ for some $\delta \geqslant 0$, show for $\mathrm{Re}(s) > \delta$ that the Dirichlet series $\sum a_n n^{-s}$ converges. In particular, when $\delta = 0$ we recover Theorem 2.45, and in fact your solution to this problem should simply involve making the appropriate adjustments to the proof of that theorem.

2. Let $f(s) = \sum a_n n^{-s}$, $g(s) = \sum b_n n^{-s}$.

   If $\sigma > \sigma_0$ is a common half-plane of absolute convergence for $f(s)$ and $g(s)$, show on this half-plane that formal multiplication is valid:

   $$\sum_{n\geqslant 1} \frac{a_n}{n^s} \cdot \sum_{n\geqslant 1} \frac{b_n}{n^s} = \sum_{n\geqslant 1} \frac{c_n}{n^s},$$

   where $c_n = \sum_{d|n} a_d \cdot b_{n/d} = \sum_{dd'=n} a_d b_{d'}$ and $\sum c_n n^{-s}$ converges absolutely on $\sigma > \sigma_0$. This result extends to products of finitely many absolutely convergent Dirichlet series.

3. a) Show the exponential of an absolutely convergent Dirichlet series is an absolutely convergent Dirichlet series. That is, if $\sum b_n n^{-s}$ converges absolutely when $\mathrm{Re}(s) > \sigma_0$, then

   $$\exp\left(\sum_{n\geqslant 1} \frac{b_n}{n^s}\right) = \sum_{n\geqslant 1} \frac{a_n}{n^s},$$

   where the right side is absolutely convergent for $\mathrm{Re}(s) > \sigma_0$.

   b) Suppose that all $b_n \geqslant 0$. Show all $a_n \geqslant 0$, and in fact $b_n \leqslant a_n$, so $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$ have the same open half-plane of absolute convergence.

4. a) Show that

$$f(s) = \sum_{n \geqslant 1} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots \Longrightarrow \lim_{\sigma \to +\infty} f(s) = a_1.$$

b) If in part a) the coefficients $a_1, \ldots, a_{k-1}$ have been determined, bring the first $k-1$ terms to the other side and consider a Dirichlet series starting with the $k$th term:

$$g(s) = \frac{a_k}{k^s} + \frac{a_{k+1}}{(k+1)^s} + \frac{a_{k+2}}{(k+2)^s} + \ldots,$$

so

$$k^s g(s) = a_k + a_{k+1} \left(\frac{k}{k+1}\right)^s + a_{k+2} \left(\frac{k}{k+2}\right)^s + \cdots = \sum_{n \geqslant k} a_n \left(\frac{k}{n}\right)^s.$$

Show that $\lim_{\sigma \to +\infty} k^s g(s) = a_k$, so all the coefficients are inductively determined by taking various limits to $\infty$.

# CHAPTER 3

# VALUES AT POSITIVE INTEGERS

## 3.1 Values of $\zeta(s)$ at Positive Even Integers

Euler became famous when he determined that

$$\zeta(2) = \frac{\pi^2}{6}.$$

The same methods led him to find $\zeta(4) = \pi^4/90$, and more generally that $\zeta(2k)$ is a rational multiple of $\pi^{2k}$ for all $k \geqslant 1$. (To this day, the so-called odd zeta-values $\zeta(3)$, $\zeta(5)$, *etc.* have no closed formula and are quite mysterious, although it is known that $\zeta(3)$ is irrational and that infinitely many more odd zeta-values irrational.)

As preparation to give Euler's formula for $\zeta(2k)$, we introduce the Bernoulli numbers.

**Definition 3.1.** The *Bernoulli numbers* $B_n$ are the coefficients in the expansion

$$\frac{x}{e^x - 1} = \sum_{n \geqslant 0} B_n \frac{x^n}{n!} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 - \frac{1}{1209600}x^8 + \dots.$$

There has not been a consistent definition of the $n$th Bernoulli number in the literature, where it may be defined as the number $B_{2n}$ instead, or perhaps as $|B_{2n}|$. Sometimes the first Bernoulli number $B_1$ has been defined as $1/2$, not $-1/2$. The value of $B_1$ will be irrelevant for the calculation of $\zeta(s)$ at positive even integers.

Aside from their uses in number theory, the Bernoulli numbers arise in other areas of mathematics. They come up in analysis in the Euler-Maclaurin summation formula and Stirling's full asymptotic asymptotic expansion for the logarithm of the Gamma function. There are also applications of Bernoulli numbers in topology.

The equation

$$x = (e^x - 1)\sum_{n\geqslant 0} B_n \frac{x^n}{n!} = \sum_{m\geqslant 1} \frac{x^m}{m!} \sum_{n\geqslant 0} \frac{B_n}{n!} x^n$$

gives the recursion

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \geqslant 2, \end{cases} \tag{3.1}$$

so the Bernoulli numbers are all rational and (3.1) gives a recursive formula for computing them.

The initial part of the series expansion for $x/(e^x-1)$ in Definition 3.1 suggests $B_n = 0$ for odd $n > 1$. Indeed,

$$\frac{x}{e^x - 1} - B_1 x = \frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^x + 1}{e^x - 1} \tag{3.2}$$

is an even function (unchanged when replacing $x$ with $-x$), so $B_n = 0$ for odd $n > 1$. Here are the first few values of the even-indexed Bernoulli numbers.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_{2k}$ | 1 | $\frac{1}{6}$ | $-\frac{1}{30}$ | $\frac{1}{42}$ | $-\frac{1}{30}$ | $\frac{5}{66}$ | $-\frac{691}{2730}$ | $\frac{7}{6}$ | $-\frac{3617}{510}$ | $\frac{43867}{798}$ | $\frac{174611}{330}$ |

Notice in particular the prime 691 in the numerator of $B_{12}$. Weil [?] refers to 691 as "a kind of tracer for Bernoulli numbers."

The initial small size of the Bernoulli numbers is misleading; they do not tend to 0, and we start to see an increase in size in the last entries of the table above. Further along, $B_{60}$ has denominator 56786730 and numerator

$$1215233140483755572040304994079820246041491.$$

Returning to Euler's calculation of $\zeta(2k)$, his motivation could be considered as the following polynomial calculation for nonzero $r_1, \ldots, r_d$:

$$f(x) = (1-r_1x)\cdots(1-r_dx) \Longrightarrow \frac{f'(x)}{f(x)} = \sum_{j=1}^{d} \frac{-r_j}{1-r_jx} = \sum_{n\geqslant 0} -(r_1^{n+1}+\cdots+r_d^{n+1})x^n$$

for small $x$ (so we can expand each $1/(1-r_jx)$ into a geometric series). The coefficients on the right are power sums. Euler applied this property of polynomials to $f(x) = \sin x$.

**Theorem 3.2 (Euler).** *For every integer $k \geqslant 1$,*

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}.$$

As a check, at $k = 1$ the theorem says $\zeta(2) = \pi^2 B_2$ and $B_2 = 1/6$. While $\zeta(2k) > 0$, the sign $(-1)^{k+1}$ in its formula is not problematic when $k$ is even (making $(-1)^{k+1} < 0$) since for such $k$ the number $B_{2k}$ is negative too: $(-1)^{k+1}B_{2k} = |B_{2k}|$ when $k$ is even.

*Proof.* The sine function has an infinite product representation

$$\sin x = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\cdots = x\prod_{n\geqslant 1}\left(1-\frac{x^2}{n^2\pi^2}\right).$$

Take the logarithmic derivative of this product to get

$$\frac{\cos x}{\sin x} = \frac{1}{x} + \sum_{n\geqslant 1}\frac{-2x}{n^2\pi^2 - x^2}.$$

Using a geometric series,

$$\frac{x\cos x}{\sin x} = 1 + \sum_{n\geqslant 1}\frac{-2x^2/n^2\pi^2}{1-x^2/n^2\pi^2} = 1 + \sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{-2x^{2m}}{n^{2m}\pi^{2m}}$$

for $|x| < \pi$. Interchanging the order of summation in this absolutely convergent double series,

$$\frac{x\cos x}{\sin x} = 1 - 2\sum_{m\geqslant 1}\frac{\zeta(2m)}{\pi^{2m}}x^{2m}. \tag{3.3}$$

On the other hand, the complex-exponential formulas for $\sin x$ and $\cos x$ yield

$$\frac{x\cos x}{\sin x} = x\frac{(e^{ix}+e^{-ix})/2}{(e^{ix}-e^{-ix})/(2i)} = ix\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = \frac{2ix}{2}\cdot\frac{e^{2ix}+1}{e^{2ix}-1},$$

so we consider the series expansion for

$$\frac{z}{2}\cdot\frac{e^z+1}{e^z-1},$$

which at $z=0$ has value 1. It is easy to see that this function is unchanged by $z\mapsto -z$, so we write

$$\frac{z}{2}\cdot\frac{e^z+1}{e^z-1} = \sum_{n\geqslant 0} b_n\frac{z^{2n}}{(2n)!}.$$

Replacing $z$ with $2iz$, we compare with (3.3) to get

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}b_k}{2(2k)!}.$$

Since

$$\frac{z}{2}\cdot\frac{e^z+1}{e^z-1} = \frac{z}{e^z-1}+\frac{z}{2},$$

we see from (3.2) that $b_k = B_{2k}$. ■

Since $\zeta(2k) > 1$, we have

$$|B_{2k}| > \frac{2(2k)!}{(2\pi)^{2k}}, \tag{3.4}$$

which tends to $\infty$ with $k$ (since $a^m/m! \to 0$ as $m\to\infty$). This proves even-indexed Bernoulli numbers get large.

In this proof, we can think of the calculation not being $\sum_{n\geqslant 1} n^{-2k}$, but being a sum over all integers,

$$\sum_{n\in\mathbf{Z}}{}' \frac{1}{n^{2k}} = 2\zeta(2k), \tag{3.5}$$

where $\sum'$ means we omit the $n=0$ term. The factor of 2 on the right side of (3.5) explains the dangling 2 in the denominator of the formula for $\zeta(2k)$.

We see here a principle that we'll meet again later: to sum over the positive integers, replace by a sum over all integers (except possibly zero). Since $\sin(\pi z)$ is an entire function which vanishes exactly at the integers, the role of $\sin x$ in the proof of Theorem 3.2 is related to the sum (3.5) being over nonzero integers.

Exercises for Section 3.1

1. From the proof of Euler's formula for $\zeta(2k)$ we obtain the Taylor expansion at $x = 0$ for the cotangent:

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{x} - \sum_{n \geqslant 1} \frac{(-1)^{n+1} 2^{2n} B_{2n}}{(2n)!} x^{2n-1}.$$

   Use the identities $\tan x = \cot x - 2\cot(2x)$ and $\csc x = \cot x + \tan(x/2)$ to get formulas for the Taylor coefficients of $\tan x$ and $\csc x$ in terms of the Bernoulli numbers. Find an expression for $\sum_{n \geqslant 0} |B_n| x^n/n!$ in terms of trigonometric functions. (Note for $k \geqslant 1$ that $|B_{2k}| = (-1)^{k+1} B_{2k} = -i^{2k} B_{2k}$.)

2. A useful symbolic notation for Bernoulli numbers is to write $B_n$ as $B^n$. For instance, the definition of the Bernoulli numbers becomes

$$\frac{x}{e^x - 1} = \sum_{n \geqslant 0} \frac{(Bx)^n}{n!} = e^{Bx}.$$

   Show the recursion (3.1) for the Bernoulli numbers becomes

$$(B+1)^m = B^m \qquad (m \geqslant 2),$$

   along with the initial value $B_1 = -1/2$ (and $B_0 = 1$).

## 3.2 Primitive Characters and Gauss Sums

We want an analogue of the evaluation of $\zeta(s)$ at positive even integers for the functions $L(s, \chi)$. To do this, and to prove other properties of $L(s, \chi)$, those Dirichlet characters satisfying a certain minimal condition are preferable. This minimal condition, called primitivity, is defined and examined in this section. The application to evaluating $L(s, \chi)$ at positive integers will come in the next section.

If $m \mid n$, then

$$a \equiv b \bmod n \Longrightarrow a \equiv b \bmod m,$$

so we get a well-defined surjection $\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/m\mathbf{Z}$, called "reduction mod $m$," sending $a$ mod $n$ to $a$ mod $m$. It is a ring homomorphism, and whenever we talk about a function $\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/m\mathbf{Z}$ where $m \mid n$, it will be understood to be this homomorphism.

As with every ring homomorphism, reduction sends units to units, so we get a group homomorphism

$$(\mathbf{Z}/(n))^{\times} \to (\mathbf{Z}/(m))^{\times}.$$

By composition with this homomorphism, every Dirichlet character mod $m$ can be lifted to the larger modulus $n$

For example, the nontrivial mod 3 character $\chi_3$ (Exercise 1.2.1) can be lifted to modulus 6, and becomes a character we denote as $\chi_6$. Their values on integers are in the table below.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_3(n)$ | 1 | −1 | 0 | 1 | −1 | 0 | 1 | −1 | 0 | 1 | −1 | 0 | 1 ... |
| $\chi_6(n)$ | 1 | 0 | 0 | 0 | −1 | 0 | 1 | 0 | 0 | 0 | −1 | 0 | 1 ... |

While $\chi_3$ and $\chi_6$ are equal as functions on integers prime to 6, they may not be equal elsewhere. Indeed, $\chi_3$ vanishes at the multiples of 3 while $\chi_6$ vanishes more often, at the multiples of 2 or 3. So their values on $\mathbf{Z}$ look different.

As another example, take the following mod 8 character:

$$\chi_8(1) = 1, \quad \chi_8(3) = -1, \quad \chi_8(5) = 1, \quad \chi_8(7) = -1. \tag{3.6}$$

Since $\chi_8(n)$ only depends on $n$ modulo 4, we see $\chi_8$ is related to the nontrivial mod 4 character $\chi_4$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_4(n)$ | 1 | 0 | −1 | 0 | 1 | 0 | −1 | 0 | 1 | 0 | −1 | 0 | 1 ... |
| $\chi_8(n)$ | 1 | 0 | −1 | 0 | 1 | 0 | −1 | 0 | 1 | 0 | −1 | 0 | 1 ... |

Note $\chi_4$ and $\chi_8$ are both equal functions on $\mathbf{Z}$, unlike $\chi_3$ and $\chi_6$, since an integer is a unit mod 8 if and only if it is a unit mod 4, namely when it is an odd integer.

**Definition 3.3.** Let $d \mid m$. A mod $m$ Dirichlet character $\chi$ is said to be *lifted* from modulus $d$ if $\chi$ is the composite of group homomorphisms

$$(\mathbf{Z}/(m))^\times \to (\mathbf{Z}/(d))^\times \to S^1,$$

where the left map is reduction mod $d$ and the right map is a Dirichlet character mod $d$.

So $\chi_6$ is a lift of $\chi_3$ to modulus 6 and $\chi_8$ is a lift of $\chi_4$ to modulus 8. Each $\mathbf{1}_m$ is a lift of $\mathbf{1}_1$. (The group $(\mathbf{Z}/(1))^\times$ is the trivial one-element group, since $\mathbf{Z}/1\mathbf{Z}$ is the zero ring, whose only element is a unit in that ring.)

If the character $\chi$ lifts to the character $\psi$, it may be the case that, as functions on $\mathbf{Z}$, $\psi$ vanishes at many more integers than $\chi$, but we will have $\psi(p) = \chi(p) \neq 0$ at all but finitely many primes $p$. So the distribution of character values at primes remains unaffected under lifting.

**Definition 3.4.** A Dirichlet character mod $m$ is called *primitive* if for every proper divisor $d$ of $m$, the character is not lifted from modulus $d$.

For example, while there is a trivial Dirichlet character for each modulus, $\mathbf{1}_1$ is the only primitive trivial character. It equals 1 at all integers (even at 0). The quadratic characters $\chi_3$ and $\chi_4$ are primitive. But the quadratic characters $\chi_6$ and $\chi_8$ are not primitive. For a prime $p$, each nontrivial character mod $p$ is primitive.

An alternate terminology for the modulus of a character is its *level*. A character is primitive if it does not come from lower level.

To prove the basic characterizing property of primitive characters (Theorem 3.6 below), we need to know that the reduction map $(\mathbf{Z}/(m))^\times \to (\mathbf{Z}/(d))^\times$ is surjective.

**Theorem 3.5.** *Let $d, m$ be integers with $d \mid m$. The natural group homomorphism*

$$(\mathbf{Z}/(m))^\times \to (\mathbf{Z}/(d))^\times$$

*is onto.*

*Proof.* Let $p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization of $m$. Let $d = p_1^{f_1} \cdots p_r^{f_r}$ be the prime factorization of $d$, so $f_j \leqslant e_j$. Using the Chinese remainder theorem, we view the reduction map $(\mathbf{Z}/(m))^\times \to (\mathbf{Z}/(d))^\times$ as the componentwise reductions

$$(\mathbf{Z}/(p_1^{e_1}))^\times \times \cdots \times (\mathbf{Z}/(p_r^{e_r}))^\times \to (\mathbf{Z}/(p_1^{f_1}))^\times \times \cdots \times (\mathbf{Z}/(p_1^{f_r}))^\times. \quad (3.7)$$

The map $(\mathbf{Z}/(p^e))^\times \to (\mathbf{Z}/(p^f))^\times$ is obviously surjective for $f = 0$. Let $f \geqslant 1$. Then every integer that is a unit mod $p^f$ is a unit mod $p^e$ since the condition for an integer to be a unit modulo $p^e$ and $p^f$ is the same, just being prime to $p$. Thus all component maps

$$(\mathbf{Z}/(p_i^{e_i}))^\times \to (\mathbf{Z}/(p_i^{f_i}))^\times$$

are onto (the unit $x$ mod $p_i^{f_i}$ is the reduction of the unit $x$ mod $p_i^{e_i}$), so the whole map (3.7) is onto. ■

The technique of reduction to the prime power case is quite handy, and is the algebraic analogue of reducing an analysis problem to the study of local power series expansions. In the above proof, what makes prime powers pleasant is that the condition for being a unit modulo different powers of a prime $p$ is the same, namely that the first base $p$ digit is not zero in $\mathbf{Z}/p\mathbf{Z}$, just as an analytic function can be locally reciprocated near a point $a$ as long as the constant term in its power series at $a$ is nonzero.

**Theorem 3.6.** *For $d \mid m$, the mod $m$ Dirichlet character $\chi$ is a lift of a character mod $d$ precisely when the character makes sense mod $d$, i.e., if and only if*

$$u, v \in (\mathbf{Z}/(m))^\times, u \equiv v \bmod d \Rightarrow \chi(u) = \chi(v).$$

*This condition is equivalent to*

$$a \in (\mathbf{Z}/(m))^\times, a \equiv 1 \bmod d \Rightarrow \chi(a) = 1.$$

*Proof.* ("only if") Suppose $\chi$ comes from a character mod $d$, say $\psi$. Then by definition,

$$(a, m) = 1 \Rightarrow \chi(a) = \psi(a).$$

In particular, if $(a, m) = 1$ and $a \equiv 1 \bmod d$ then $\chi(a) = 1$.

("if") To show the converse, we will use the given implication to *construct* a mod $d$ character $\psi$ that lifts to $\chi$. Let $j$ be prime to $d$, *i.e.*, $j \in (\mathbf{Z}/(d))^\times$. How do we define $\psi(j)$? While $j$ may not be prime to $m$, by Theorem 3.5 there is some $j'$ prime to $m$ with $j' \equiv j \bmod d$. So if $\chi$ comes from a mod $d$ character $\psi$, it must be the case that

$$\psi(j) = \psi(j') = \chi(j').$$

This gives us a defining recipe for $\psi$: associate to a unit class $j \bmod d$ a unit class $j' \bmod m$, where $j' \equiv j \bmod d$, and evaluate $\chi(j')$. Is this definition of $\psi$ well-defined? Suppose $j', j''$ are both units mod $m$ that reduce to the same class mod $d$. Write $j' \equiv j''k \bmod m$ for some unit $k \bmod m$. Reducing mod $d$, we get $k \equiv 1 \bmod d$, so

$$\chi(j') = \chi(j'')\chi(k) = \chi(j'')$$

by the "if" direction hypothesis. Thus $\psi$ is a well-defined character mod $d$. ■

So the obstruction to a character mod $m$ being primitive is that it is trivial on the units mod $m$ that reduce to 1 modulo some proper divisor of $m$. We formulate this as

**Corollary 3.7.** *A mod $m$ Dirichlet character $\chi$ is primitive when for each proper divisor $d$ of $m$ there is some $u \in (\mathbf{Z}/(m))^\times$ such that $u \equiv 1 \bmod d$ and $\chi(u) \neq 1$. In other words, $\chi$ is primitive when, for each proper divisor $d$ of $m$, the kernel of $\chi$ does not contain the kernel of the reduction map $(\mathbf{Z}/(m))^\times \to (\mathbf{Z}/(d))^\times$.*

In practice, the only divisors $d$ that we have to consider in Corollary 3.7 are the maximal divisors $m/p$ for primes $p$, because if the criterion applies to these particular divisors of $m$ then it applies to all proper divisors of $m$.

It may seem that Theorem 3.6 is saying a mod $m$ character comes from modulus $d$ precisely when it has period $d$, so if a mod $m$ character has minimal period $m$ on $\mathbf{Z}$, then it is primitive. This is a simpler test than Corollary 3.7, and it is given as a description of primitivity in some books. It is also sometimes false! For example, $\chi_6$, as a function on $\mathbf{Z}$, has minimal period 6 but is not primitive. Because a character defined modulo a prime power defines the same function on $\mathbf{Z}$ as any of its lifts to a higher power of that prime (consider $\chi_4$ and $\chi_8$), the simple primitivity test does work for characters modulo prime powers: a Dirichlet character mod $p^e$ is primitive exactly when it has minimal period $p^e$ as a function on the integers.

**Definition 3.8.** The *conductor* of a mod $m$ Dirichlet character is the smallest (positive) divisor $f$ of $m$ for which the character is a lift from modulus $f$.

For example, every trivial Dirichlet character has conductor 1. The conductor of $\chi_6$ is 3. The conductor of $\chi_3$ is 3. For a prime $p$, the conductor of a nontrivial character mod $p$ is $p$. The conductor of any character mod $p^k$ is its minimal period as a function on $\mathbf{Z}$. A character and its conjugate have the same conductor: $f_\chi = f_{\overline{\chi}}$. The notation $f = f_\chi$ for the conductor comes from German (Führer). We will usually write $\chi_*$ for the mod $f$ primitive character corresponding to (or, as we may say, attached to) $\chi$.

**Theorem 3.9.** *Let $\zeta$ be a primitive $m$th root of unity.*

*a) For $j \in \mathbf{Z}$, $\zeta^j = 1$ precisely when $m \mid j$.*

*b) For $j \in \mathbf{Z}$, the order of $\zeta^j$ is $m/(j,m)$. In particular, $\zeta^j$ has order $m$ precisely when $(j,m) = 1$.*

*Proof.* Exercise. ∎

Intuitively, part b) should say that if $\zeta$ has order $m$ then $\zeta^j$ has order $m/j$. The problem is that $j$ may not be a factor of $m$. So the actual order is $m$ divided by the largest piece of $j$ that is a factor of $m$, namely by $(m,j)$. If $j \mid m$, then $\zeta^j$ does have order $m/j$.

How are the $L$-functions of a Dirichlet character and its lifts related?

**Example 3.10.** We compare $\chi_3$ and $\chi_6$:

$$L(s,\chi_3) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{10^s} - \frac{1}{11^s} + \frac{1}{13^s} - \dots$$

and

$$L(s,\chi_6) = 1 - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{11^s} + \frac{1}{13^s} - \frac{1}{17^s} + \frac{1}{19^s} - \frac{1}{23^s} + \frac{1}{29^s} - \dots,$$

so

$$\begin{aligned} L(s,\chi_6) - L(s,\chi_3) &= \frac{1}{2^s} - \frac{1}{4^s} + \frac{1}{8^s} - \frac{1}{10^s} + \dots \\ &= \frac{1}{2^s}\left(1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \dots\right) \\ &= \frac{1}{2^s} L(s,\chi_3). \end{aligned}$$

We have

$$L(s,\chi_6)=\left(1+\frac{1}{2^s}\right)L(s,\chi_3).$$

**Example 3.11.** Since $\chi_4$ and $\chi_8$ are the same function on $\mathbf{Z}$, $L(s,\chi_4)=L(s,\chi_8)$.

**Example 3.12.** $\mathbf{1}_1$ and $\mathbf{1}_m$. We have

$$L(s,\mathbf{1}_m)=\sum_{(n,m)=1}\frac{1}{n^s}=\prod_{(p,m)=1}\frac{1}{1-p^{-s}}=\zeta(s)\cdot\prod_{p|m}\left(1-\frac{1}{p^s}\right),$$

so $L(s,\mathbf{1}_m)$ is the zeta-function with Euler factors corresponding to the primes dividing $m$ removed. Then $\zeta(s)=L(s,\mathbf{1}_1)$ and

$$L(s,\mathbf{1}_m)=\prod_{p|m}\left(1-\frac{1}{p^s}\right)\cdot L(s,\mathbf{1}_1).$$

**Theorem 3.13.** *Let $\chi$ be a mod $m$ Dirichlet character with conductor $f$, $\chi_*$ the corresponding mod $f$ primitive Dirichlet character. For $\sigma>1$,*

$$L(s,\chi)=\prod_{\substack{p|m\\(p,f)=1}}\left(1-\frac{\chi_*(p)}{p^s}\right)\cdot L(s,\chi_*).$$

*So $L(s,\chi)=L(s,\chi_*)$ if and only if $m$ and $f$ have the same prime factors.*

*Proof.* We compare the Euler products, not the Dirichlet series. The Euler product for $L(s,\chi)$ is

$$\prod_{(p,m)=1}\frac{1}{1-\chi(p)p^{-s}}=\prod_{(p,m)=1}\frac{1}{1-\chi_*(p)p^{-s}}.$$

The Euler product for $L(s,\chi_*)$ is

$$\begin{aligned}\prod_{(p,f)=1}\frac{1}{1-\chi_*(p)p^{-s}} &= \prod_{\substack{p|m\\(p,f)=1}}\frac{1}{1-\chi_*(p)p^{-s}}\cdot\prod_{(p,m)=1}\frac{1}{1-\chi_*(p)p^{-s}}\\ &= \prod_{\substack{p|m\\(p,f)=1}}\frac{1}{1-\chi_*(p)p^{-s}}\cdot L(s,\chi).\end{aligned}$$

■

Sums of roots of unity, also called exponential sums, are an important tool in number theory. To compute $L(s,\chi)$ on positive integers we'll meet the following exponential sum, whose properties are worth tabulating in advance.

**Definition 3.14.** Let $\chi$ be a Dirichlet character mod $m$, $\zeta$ an $m$th root of unity in $\mathbf{C}$. The *Gauss sum* corresponding to $\chi$ and $\zeta$ is

$$G(\chi,\zeta) := \sum_{j\in\mathbf{Z}/m\mathbf{Z}} \chi(j)\zeta^j = \sum_{j\in(\mathbf{Z}/(m))^\times} \chi(j)\zeta^j.$$

So a Gauss sum is a sum of roots of unity. The Gauss sum need only have $j$ run over the units mod $m$, but it will be convenient to let $j$ run over all of $\mathbf{Z}/m\mathbf{Z}$ sometimes. We denote the complex conjugate of $G(\chi,\zeta)$ as $\overline{G}(\chi,\zeta)$.

**Definition 3.15.** For a mod $m$ Dirichlet character $\chi$, its *standard Gauss sum* is $G(\chi) \equiv G(\chi, e^{2\pi i/m})$.

The terminology "standard" is not standard. One aspect of the notation $G(\chi)$ to be attentive to is that it obscures the definite choice of root of unity $e^{2\pi i/m}$. For example,

$$G((\tfrac{\cdot}{5}), e^{2\pi i/5}) = G((\tfrac{\cdot}{5}), e^{8\pi i/5}) = e^{2\pi i/5} - e^{4\pi i/5} - e^{6\pi i/5} + e^{8\pi i/5} = \sqrt{5},$$

$$G((\tfrac{\cdot}{5}), e^{4\pi i/5}) = G((\tfrac{\cdot}{5}), e^{6\pi i/5}) = e^{4\pi i/5} - e^{8\pi i/5} - e^{2\pi i/5} + e^{6\pi i/5} = -\sqrt{5}.$$

So the choice of root of unity affects the value of the Gauss sum. (These Gauss sum evaluations will follow from later results in this section, if the reader does not want to check them by hand.) In the initial theory of Gauss sums, we do not even need $\zeta$ to be a root of unity of exact order $m$. It simply has to satisfy $\zeta^m = 1$, so that $\zeta^j$ makes sense for $j \in \mathbf{Z}/m\mathbf{Z}$. Note $G(\chi, 1) = 0$ for nontrivial $\chi$.

**Example 3.16.** Let's look at a Gauss sum for the quadratic character mod 17. Since $(\frac{-1}{17}) = 1$, $(\frac{a}{17})$ is even. In the table are the first half of the values.

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $(\frac{j}{17})$ | 0 | 1 | 1 | $-1$ | 1 | $-1$ | $-1$ | $-1$ | 1 |

$$G_{17} = \sum_{j=1}^{16} \left(\frac{j}{17}\right) e^{2\pi i j/17} = 4.123105625617\ldots,$$

so the value of $G_{17}$ is much smaller than the naive bound $|G_{17}| \leqslant 16$, and in fact $G_{17} = \sqrt{17}$. There is a lot of cancellation.

**Theorem 3.17.** *For a prime $p$, a nontrivial mod $p$ Dirichlet character $\chi$, and a nontrivial $p$th root of unity $\zeta$, $|G(\chi,\zeta)| = \sqrt{p}$.*

*Proof.* We will prove the result in the form $G(\chi,\zeta)\overline{G}(\chi,\zeta) = p$. The product $G(\chi,\zeta)\overline{G}(\chi,\zeta)$ equals

$$\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\chi(j)\overline{\chi}(k)\zeta^{j-k} = \sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\chi(j)\overline{\chi}(jk)\zeta^{j-jk} = \sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\overline{\chi}(k)\zeta^{j-jk}.$$

Interchanging the sums,

$$\begin{aligned} G(\chi,\zeta)\overline{G}(\chi,\zeta) &= \sum_{k=1}^{p-1}\overline{\chi}(k)\left(\sum_{j=1}^{p-1}\zeta^{j(1-k)}\right) \\ &= p-1-\sum_{k=2}^{p-1}\overline{\chi}(k) \quad \text{by Exercise 3.2.1} \\ &= p-\sum_{k=1}^{p-1}\overline{\chi}(k) \\ &= p. \end{aligned}$$

■

Since

$$\overline{G}(\chi,\zeta) = \sum_{j\in(\mathbf{Z}/(p))^\times}\overline{\chi}(j)\zeta^{-j} = \overline{\chi}(-1)G(\overline{\chi},\zeta) = \chi(-1)G(\overline{\chi},\zeta), \tag{3.8}$$

we can rewrite Theorem 3.17 as

$$G(\chi,\zeta)G(\overline{\chi},\zeta) = \chi(-1)p.$$

Suppose $p \neq 2$ and let $\chi$ be the Legendre symbol $(\frac{\cdot}{p})$. Since $\overline{\chi} = \chi$, we get

$$G((\tfrac{\cdot}{p}),\zeta)^2 = (\tfrac{-1}{p})p. \tag{3.9}$$

For instance, if $p = 17$ then $(\frac{-1}{17}) = 1$ and we find the Gauss sum squares to 17; being a real number that is calculated (in Example 3.16) to be positive, it must be $\sqrt{17}$.

**Theorem 3.18.** *If $\chi$ is primitive mod $m$ and the $m$th root of unity $\zeta$ has order less than $m$, then $G(\chi,\zeta) = 0$.*

*Proof.* Let $\zeta$ have order $m'$, so $m' \mid m$ and $m' < m$.

Since $\chi$ is primitive, there is a unit $c$ mod $m$ with $c \equiv 1 \bmod m'$ and $\chi(c) \neq 1$. (If there weren't such $c$, then $\chi$ comes from modulus $m'$ by Theorem 3.6.)

Let $bc \equiv 1 \bmod m$. So $b \equiv 1 \bmod m'$. Then $\zeta^b = \zeta$, so

$$\chi(c)G(\chi,\zeta) = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \chi(cj)\zeta^j = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \chi(j)\zeta^{bj} = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \chi(j)\zeta^j = G(\chi,\zeta).$$

Since $\chi(c) \neq 1$, $G(\chi,\zeta) = 0$. ■

As a special case of Theorem 3.18, taking $\zeta = 1$, we have

$$0 = G(\chi, 1) = \sum_{j \bmod m} \chi(j)$$

for a primitive character $\chi$ mod $m$. This particular formula is true for all nontrivial characters mod $m$, primitive or not, by the reasoning used in the above proof. If $\chi$ is nontrivial pick $c$ mod $m$ so that $(c, m) = 1$ and $\chi(c) \neq 1$. Then, letting $S = \sum_{j \bmod m} \chi(j)$, we have

$$\chi(c)S = \chi(c) \sum_{j \bmod m} \chi(j) = \sum_{j \bmod m} \chi(c)\chi(j) = \sum_{j \bmod m} \chi(cj) = \sum_{j \bmod m} \chi(j) = S$$

since $j \bmod m \mapsto cj \bmod m$ is a permutation of $\mathbf{Z}/m\mathbf{Z}$.

Here is the key property of primitive characters.

**Theorem 3.19.** *Let $\chi$ be a primitive character mod $m$ and $\zeta$ be a primitive $m$th root of unity. For every $n \in \mathbf{Z}$,*

$$G(\chi, \zeta^n) = \overline{\chi}(n)G(\chi,\zeta).$$

If $\chi$ is not primitive, this identity breaks down at some integers. See Exercise 3.2.6.

*Proof.* First suppose $(m, n) = 1$, so $\zeta^n$ has order $m$. Then

$$G(\chi,\zeta^n) = \sum_{j \bmod m} \chi(j)\zeta^{nj} = \sum_{j \bmod m} \chi(jn^{-1})\zeta^j = \overline{\chi}(n)G(\chi,\zeta).$$

If $\gcd(m, n) > 1$, $\zeta^n$ has order less than $m$, so $G(\chi,\zeta^n) = 0$ by Theorem 3.18 and $\overline{\chi}(n)G(\chi,\zeta) = 0$ by the *definition* of Dirichlet characters at integers not prime to the modulus. ■

**Theorem 3.20.** *If $\chi$ is primitive mod $m$ and $\zeta$ is an $m$th root of unity, then*

$$G(\chi,\zeta)G(\overline{\chi},\zeta) = \begin{cases} \chi(-1)m, & \text{if } \zeta \text{ has order } m, \\ 0, & \text{if } \zeta \text{ has order less than } m. \end{cases}$$

*In particular, if $\zeta$ has order $m$ then $|G(\chi,\zeta)| = \sqrt{m}$.*

Note $\chi(-1) = \pm 1$ since $\chi(-1)^2 = \chi((-1)^2) = \chi(1) = 1$.

*Proof.* By Theorem 3.18, we may assume $\zeta$ has order $m$. In that case the identity we want to show is the same as $G(\chi,\zeta)\overline{G}(\chi,\zeta) = m$. The product of $G(\chi,\zeta)$ and its conjugate equals

$$\begin{aligned}
\sum_{j\in\mathbf{Z}/m}\sum_{k\in\mathbf{Z}/m}\chi(j)\overline{\chi}(k)\zeta^{j-k} &= \sum_{j\in(\mathbf{Z}/m)^\times}\sum_{k\in\mathbf{Z}/m}\chi(j)\overline{\chi}(jk)\zeta^{j-jk} \\
&= \sum_{j\in(\mathbf{Z}/m)^\times}\sum_{k\in\mathbf{Z}/m}\overline{\chi}(k)\zeta^{-jk}\zeta^{j} \\
&= \sum_{j\in(\mathbf{Z}/m)^\times} G(\overline{\chi},\zeta^{-j})\zeta^{j} \\
&= \sum_{j\in\mathbf{Z}/m} G(\overline{\chi},\zeta^{-j})\zeta^{j} \qquad \text{by Theorem 3.18} \\
&= \sum_{j\in\mathbf{Z}/m}\sum_{k\in\mathbf{Z}/m}\overline{\chi}(k)\zeta^{-jk}\zeta^{j} \\
&= \sum_{k\in\mathbf{Z}/m}\sum_{j\in\mathbf{Z}/m}\overline{\chi}(k)\zeta^{j(1-k)} \\
&= \overline{\chi}(1)m \\
&= m.
\end{aligned}$$

■

Exercises for Section 3.2

1. Let $\zeta$ be a root of unity of order $m$. Show

$$\sum_{j=0}^{m-1}\zeta^{jk} = \begin{cases} m, & \text{if } k \equiv 0 \bmod m, \\ 0, & \text{if } k \not\equiv 0 \bmod m. \end{cases}$$

2. Prove Theorem 3.9.

3. Show there is no primitive character modulo 2 or 6 or, more generally, $2N$ where $N$ is odd.

4. a) Make a list of all the Dirichlet characters mod 12 and determine the conductor of each one (there are four characters).

   b) Make a list of all the Dirichlet characters mod 9 and determine the conductor of each one (there are six characters, and if $\omega$ is a nontrivial cube root of unity then $-\omega$ has order 6).

5. By the Chinese remainder theorem, $(\mathbf{Z}/(36))^\times \cong (\mathbf{Z}/(4))^\times \times (\mathbf{Z}/(9))^\times$. The first factor is cyclic with generator 3 mod 4 and the second is cyclic with generator 2 mod 9.

   a) Solve the congruences $x \equiv 3 \bmod 4$, $x \equiv 1 \bmod 9$ and $x \equiv 1 \bmod 4$, $x \equiv 2 \bmod 9$. Verify that the resulting numbers together generate $(\mathbf{Z}/(36))^\times$.

   b) Show the following function is *not* a Dirichlet character mod 36.

| $n$ | 1 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 25 | 29 | 31 | 35 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F(n)$ | 1 | 1 | $-1$ | $-1$ | $-1$ | 1 | $-1$ | 1 | 1 | 1 | $-1$ | $-1$ |

   c) Compute the conductors of each of the following Dirichlet characters mod 36.

| $n$ | 1 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 25 | 29 | 31 | 35 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_a(n)$ | 1 | $-1$ | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_b(n)$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_c(n)$ | 1 | $\omega^2$ | $\omega$ | $\omega$ | $\omega^2$ | 1 | 1 | $\omega^2$ | $\omega$ | $\omega$ | $\omega^2$ | 1 |

6. Let $\chi$ be a character mod $m$ that is *not* primitive and $f$ be its conductor, so $f \mid m$. For $\zeta = e^{2\pi i/m}$, show the identity

$$G(\chi, \zeta^n) = \overline{\chi}(n)G(\chi, \zeta).$$

   from Theorem 3.19 (a theorem about primitive characters!) is *false* when $n = m/f$: the left side is not 0 and the right side is 0.

## 3.3 Values of $L(s,\chi)$ at Positive Even or Odd Integers

Generalizing what we saw about $\zeta(s)$ on even numbers, we will find a formula for $L(s,\chi)$ either when $s$ is a positive even integer or a positive odd integer. By Theorem 3.13, $L(s,\chi)$ can be expressed in terms of $L(s,\chi_*)$ where $\chi_*$ is the primitive character associated to $\chi$, so we will focus on primitive nontrivial $\chi$ (so $m \geqslant 3$ – see Exercise 3.2.3).

**Theorem 3.21.** *For a primitive Dirichlet character* $\chi \bmod m$ *and all* $n \in \mathbf{Z}$*,*

$$\chi(n) = \frac{\chi(-1)G(\chi)}{m} G(\overline{\chi}, e^{2\pi i n/m}).$$

*where* $G(\chi) = G(\chi, e^{2\pi i/m}) = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \chi(j) e^{2\pi i j/m}$.

*Proof.* Since $\chi$ is primitive, by Theorem 3.19 we have for each $n \in \mathbf{Z}$

$$G(\overline{\chi}, e^{2\pi i n/m}) = \overline{\overline{\chi}}(n) G(\overline{\chi}, e^{2\pi i/m}) = \chi(n) G(\overline{\chi}),$$

By Theorem 3.20, $G(\chi)G(\overline{\chi}) = \chi(-1)m$, so $G(\overline{\chi}, e^{2\pi i n/m}) = \dfrac{\chi(n)\chi(-1)m}{G(\chi)}$. Thus

$$\chi(n) = \frac{G(\chi)}{m\chi(-1)} G(\overline{\chi}, e^{2\pi i n/m}).$$

Since $\chi(-1) = \pm 1$, we can move that term into the numerator. ∎

For $\mathrm{Re}(s) > 1$, the formula for $\chi(n)$ in Theorem 3.21 gives us

$$\begin{aligned}
L(s,\chi) &= \sum_{n \geqslant 1} \frac{\chi(n)}{n^s} \\
&= \sum_{n \geqslant 1} \frac{\chi(-1)G(\chi)}{m} \frac{G(\overline{\chi}, e^{2\pi i n/m})}{n^s} \\
&= \frac{\chi(-1)G(\chi)}{m} \sum_{n \geqslant 1} \frac{G(\overline{\chi}, e^{2\pi i n/m})}{n^s} \\
&= \frac{\chi(-1)G(\chi)}{m} \sum_{n \geqslant 1} \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \frac{\overline{\chi}(j) e^{2\pi i j n/m}}{n^s} \\
&= \frac{\chi(-1)G(\chi)}{m} \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \overline{\chi}(j) \left( \sum_{n \geqslant 1} \frac{e^{2\pi i j n/m}}{n^s} \right).
\end{aligned}$$

Setting $s$ to be an integer $k \geqslant 2$,

$$L(k,\chi) = \frac{\chi(-1)G(\chi)}{m} \sum_{j \in \mathbf{Z}/m\mathbf{Z}} \overline{\chi}(j) \left( \sum_{n \geqslant 1} \frac{e^{2\pi i j n/m}}{n^k} \right). \tag{3.10}$$

The term in the summation at $j = 0$ can be omitted since $\chi(0) = 0$. (Our modulus $m$ is at least 3 since our $\chi$ is a nontrivial primitive character.)

Inspired by the inner sum in (3.10), for any positive integer $k \geqslant 1$ and real $x$ we define

$$A_k(x) := {\sum_{n \in \mathbf{Z}}}' \frac{e^{2\pi i n x}}{n^k}.$$

where the $'$ means we omit $n = 0$. When $k \geqslant 2$, the series is absolutely and uniformly convergent on $\mathbf{R}$, so it is continuous. This series is not absolutely convergent when $k = 1$, so $A_1(x)$ is more subtle.

Note the sum in $A_k(x)$, in contrast to the inner series in (3.10), runs over all nonzero integers rather than the positive integers. We will see that $A_k(x)$ is a much simpler function of $x$ than its definition as a series suggests, at least when $0 \leqslant x \leqslant 1$.

For $k \geqslant 2$, we can combine terms at $n$ and $-n$ in $A_k(x)$ together by absolute convergence, getting

$$A_k(x) = \sum_{n \geqslant 1} \left( \frac{e^{2\pi i n x}}{n^k} + \frac{e^{-2\pi i n x}}{(-n)^k} \right) = \sum_{n \geqslant 1} \frac{e^{2\pi i n x} + (-1)^k e^{-2\pi i n x}}{n^k}.$$

Setting $x = 0$ in this formula,

$$A_k(0) = \begin{cases} 2\zeta(k), & \text{for } k \geqslant 2 \text{ even}, \\ 0, & \text{for } k \geqslant 3 \text{ odd}. \end{cases} \tag{3.11}$$

When $k = 2$ we have $A_2(x) = 2\sum_{n \geqslant 1} \frac{\cos(2\pi n x)}{n^2}$. A standard result from Fourier analysis says that for $0 \leqslant \theta \leqslant 2\pi$,

$$\sum_{n \geqslant 1} \frac{\cos(n\theta)}{n^2} = \frac{(\theta - \pi)^2}{4} - \frac{\pi^2}{12} \Longrightarrow 2\sum_{n \geqslant 1} \frac{\cos(n\theta)}{n^2} = \frac{(\theta - \pi)^2}{2} - \frac{\pi^2}{6}$$

Setting $\theta = 2\pi x$ for $0 \leqslant x \leqslant 1$,

$$A_2(x) = 2\sum_{n\geqslant 1} \frac{\cos(2\pi n x)}{n^2} = \frac{(2\pi x - \pi)^2}{2} - \frac{\pi^2}{6} = 2\pi^2\left(x^2 - x + \frac{1}{6}\right), \qquad (3.12)$$

so $A_2(x)$ is a polynomial in $x$ when $x \in [0,1]$. Since $A_2(x+1) = A_2(x)$, $A_2(x)$ as a function on $\mathbf{R}$ is the periodic extension of the polynomial $\pi^2(x^2 - x + 1/6)$ from $[0,1]$ to $\mathbf{R}$.

Termwise differentiation of $A_k(x)$ when $k \geqslant 3$ can be justified, leading to

$$\frac{d}{dx}A_k(x) = 2\pi i A_{k-1}(x) \qquad (3.13)$$

for all $x$ when $k \geqslant 3$. For example, when $0 \leqslant x \leqslant 1$,

$$A_3'(x) = 2\pi i A_2(x) = 4\pi^3 i\left(x^2 - x + \frac{1}{6}\right),$$

so for $0 \leqslant x \leqslant 1$,

$$A_3(x) = 4\pi^3 i\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6} + C\right).$$

Since $A_3(0) = 0$ by (3.11), we have $C = 0$, so for $0 \leqslant x \leqslant 1$,

$$A_3(x) = \frac{4\pi^3 i}{3}\left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right).$$

Next, from $A_4'(x) = 2\pi i A_3(x)$, for $0 \leqslant x \leqslant 1$ we have

$$A_4'(x) = -\frac{8\pi^4}{3}\left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right),$$

so

$$A_4(x) = -\frac{8\pi^4}{3}\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4} + C\right)$$

when $0 \leqslant x \leqslant 1$. Setting $x = 0$ here, (3.11) gives us

$$2\frac{\pi^4}{90} = -\frac{8\pi^4}{3}C \Longrightarrow C = -\frac{6}{720} = -\frac{1}{120},$$

so for $0 \leqslant x \leqslant 1$,

$$A_4(x) = -\frac{8\pi^4}{3}\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4} - \frac{1}{120}\right) = -\frac{(2\pi)^4}{24}\left(x^4 - 2x^3 + x^2 - \frac{1}{30}\right).$$

By induction, for $k \geqslant 2$ and $0 \leqslant x \leqslant 1$, $A_k(x)$ equals $-(2\pi i)^k/k!$ times a monic polynomial of degree $k$ with rational coefficients. Call that polynomial $\mathbf{B}_k(x)$, so based on our calculations of $A_k(x)$ for $k = 2, 3, 4$ we have

$$\mathbf{B}_2(x) = x^2 - x + \frac{1}{6}, \quad \mathbf{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}, \quad \mathbf{B}_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

For $k \geqslant 3$, the condition $A_k'(x) = 2\pi i A_{k-1}(x)$ is the same as

$$\left(-\frac{(2\pi i)^k}{k!}\mathbf{B}_k(x)\right)' = 2\pi i\left(-\frac{(2\pi i)^{k-1}}{(k-1)!}\mathbf{B}_{k-1}(x)\right),$$

which after simplifying is equivalent to

$$\mathbf{B}_k'(x) = k\mathbf{B}_{k-1}(x) \tag{3.14}$$

for $0 \leqslant x \leqslant 1$, and thus for all $x \in \mathbf{R}$ since polynomials equal on $[0,1]$ are equal everywhere. Armed with (3.14) and the initial value $\mathbf{B}_2(x) = x^2 - x + 1/6$, we can recursively solve for each $\mathbf{B}_k(x)$ provided we know their constant terms. Let's evaluate $\mathbf{B}_k(0)$.

By (3.11), the constant term of $\mathbf{B}_k(x)$ is

$$\mathbf{B}_k(0) = -\frac{k!}{(2\pi i)^k}A_k(0) = \begin{cases} -\dfrac{k!}{(2\pi i)^k}(2\zeta(k)), & \text{for } k \geqslant 2 \text{ even,} \\ 0, & \text{for } k \geqslant 3 \text{ odd.} \end{cases}$$

Theorem 3.2 tells us $\zeta(s)$ at even integers: if $k = 2\ell$ then

$$\mathbf{B}_{2\ell}(0) = -\frac{(2\ell)!}{(2\pi i)^{2\ell}}2\zeta(2\ell) = -\frac{(2\ell)!}{(2\pi)^{2\ell}(-1)^{\ell}}2\frac{(-1)^{\ell+1}(2\pi)^{2\ell}B_{2\ell}}{2(2\ell)!} = B_{2\ell},$$

so $\mathbf{B}_k(0) = B_k$ when $k \geqslant 2$ is even. When $k \geqslant 3$ is odd, then $\mathbf{B}_k(0) = 0$ and we also know $B_k = 0$, so

$$\mathbf{B}_k(0) = B_k \text{ for all } k \geqslant 2.$$

What about the case $k = 1$? Let's define $\mathbf{B}_1(x)$ so that $\mathbf{B}_k'(x) = k\mathbf{B}_{k-1}(x)$ at $k = 2$: $(x^2 - x + 1/6)' = 2\mathbf{B}_1(x)$. This makes $\mathbf{B}_1(x) = x - 1/2$, so $\mathbf{B}_1(0) = -1/2 = B_1$. And define $\mathbf{B}_0(x)$ so that $\mathbf{B}_k'(x) = k\mathbf{B}_{k-1}(x)$ at $k = 1$: $\mathbf{B}_0(x) = 1 = B_0$. Thus

$$\mathbf{B}_k(0) = B_k \text{ for all } k \geqslant 0. \tag{3.15}$$

Let's find the coefficients of $\mathbf{B}_k(x)$. If we write $\mathbf{B}_k(x) = \sum_{j=0}^{k} b_{j,k}x^j$ then $b_{j,k} = \mathbf{B}_k^{(j)}(0)/j!$. The derivative relation $\mathbf{B}_k'(x) = k\mathbf{B}_{k-1}(x)$, after iterating it once, implies

$$\mathbf{B}_k''(x) = k(k-1)\mathbf{B}_{k-2}(x)$$

and more generally its $j$th derivative is

$$\mathbf{B}_k^{(j)}(x) = k(k-1)\cdots(k-(j-1))\mathbf{B}_{k-j}(x)$$

for $j \leqslant k$ so

$$b_{j,k} = \frac{\mathbf{B}_k^{(j)}(0)}{j!} = \frac{k(k-1)\dots(k-j+1)}{j!}\mathbf{B}_{k-j}(0) = \binom{k}{j}B_{k-j}$$

and thus

$$\mathbf{B}_k(x) = \sum_{j=0}^{k}\binom{k}{j}B_{k-j}x^j = \sum_{j=0}^{k}\binom{k}{j}B_j x^{k-j}.$$

Now we have a formula for $A_k(x) = \sum_{n\in\mathbf{Z}}' e^{2\pi inx}/n^k$ when $0 \leqslant x \leqslant 1$, in terms of Bernoulli numbers:

$$k \geqslant 2 \Longrightarrow A_k(x) = -\frac{(2\pi i)^k}{k!}\mathbf{B}_k(x) = -\frac{(2\pi i)^k}{k!}\sum_{j=0}^{k} B_{k-j}x^j. \tag{3.16}$$

What happens at $k = 1$? Regard $A_1(x) = \sum_{n\in\mathbf{Z}}' e^{2\pi inx}/n$ as having terms at $n$ and $-n$ put together: $\sum_{n\geqslant 1}(e^{2\pi inx}/n + e^{-2\pi inx}/(-n)) = 2i\sum_{n\geqslant 1}\sin(2\pi nx)/n$. By Fourier analysis it can be shown that

$$\sum_{n\geqslant 1}\frac{\sin(2\pi nx)}{n} = \frac{\pi - x}{2} = -\pi\left(x - \frac{1}{2}\right)$$

for $0 < x < 1$ (*not* at $x = 0$ or $1$), so

$$0 < x < 1 \Longrightarrow A_1(x) = -2\pi i\left(x - \frac{1}{2}\right) = -2\pi i\sum_{j=0}^{1} B_{1-j}x^j.$$

This means the formula for $A_k(x)$ in (3.16) is valid for $k \geqslant 1$, not just $k \geqslant 2$.

Returning to the computation of $L(k,\chi)$ in (3.10), we want to relate

$$\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}}{n^k} \quad \text{and} \quad A_k(j/m)=\sum_{n\neq 0}\frac{e^{2\pi ijn/m}}{n^k}.$$

In (3.10) the terms at $j \bmod m$ and $-j \bmod m$ together.[1]

$$\overline{\chi}(j)\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}}{n^k}+\overline{\chi}(-j)\sum_{n\geqslant 1}\frac{e^{-2\pi ijn/m}}{n^k}=\overline{\chi}(j)\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}+\chi(-1)e^{-2\pi ijn/m}}{n^k}.$$

The number $\chi(-1)$ is $\pm 1$. *If* this sign equals $(-1)^k$ then we are in good shape, because we get $A_k(j/m)$:

$$\begin{aligned}\overline{\chi}(j)\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}+\chi(-1)e^{-2\pi ijn/m}}{n^k} &= \overline{\chi}(j)\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}+(-1)^k e^{-2\pi ijn/m}}{n^k}\\ &= \overline{\chi}(j)A_k(j/m).\end{aligned}$$

Since a sum over $j \bmod m$ and $-j \bmod m$ will be equal, (3.10) becomes

$$\begin{aligned}L(k,\chi) &= \frac{\chi(-1)G(\chi)}{m}\sum_{j\in\mathbf{Z}/m\mathbf{Z}}\overline{\chi}(j)\left(\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}}{n^k}\right)\\ &= \frac{\chi(-1)G(\chi)}{2m}\left(\sum_{j\in\mathbf{Z}/m\mathbf{Z}}\overline{\chi}(j)\left(\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}}{n^k}\right)+\sum_{j\in\mathbf{Z}/m\mathbf{Z}}\overline{\chi}(-j)\left(\sum_{n\geqslant 1}\frac{e^{-2\pi ijn/m}}{n^k}\right)\right)\\ &= \frac{\chi(-1)G(\chi)}{2m}\sum_{j\in\mathbf{Z}/m\mathbf{Z}}\overline{\chi}(j)\left(\sum_{n\geqslant 1}\frac{e^{2\pi ijn/m}+\chi(-1)e^{-2\pi ijn/m}}{n^k}\right)\\ &= \frac{\chi(-1)G(\chi)}{2m}\sum_{j\in\mathbf{Z}/m\mathbf{Z}}\overline{\chi}(j)A_k(j/m)\\ &= \frac{\chi(-1)G(\chi)}{2m}\sum_{j=1}^{m-1}\overline{\chi}(j)\left(\frac{-(2\pi i)^k}{k!}\right)\mathbf{B}_k\left(\frac{j}{m}\right)\\ &= \frac{-\chi(-1)G(\chi)(2\pi i)^k}{2m\cdot k!}\sum_{j=1}^{m-1}\overline{\chi}(j)\mathbf{B}_k\left(\frac{j}{m}\right) \qquad (3.17)\end{aligned}$$

Let's summarize what we've found.

**Theorem 3.22.** *If $\chi$ is a primitive character mod $m$ and $k$ is a positive integer*

[1] If $\chi(j)\neq 0$, so $(j,m)=1$, then $j\not\equiv -j \bmod m$ since $m>2$.

*such that* $\chi(-1) = (-1)^k$ *then*

$$L(k,\chi) = \frac{-\chi(-1)G(\chi)(2\pi i)^k}{2m \cdot k!} \sum_{j=1}^{m-1} \overline{\chi}(j)\mathbf{B}_k\left(\frac{j}{m}\right),$$

*where* $G(\chi) = \sum_{a \bmod m} \chi(a)e^{2\pi i a/m}$ *and* $\mathbf{B}_k(x) = \sum_{j=0}^{k} \binom{k}{\ell} B_j x^{k-j}$.

This theorem has a parity condition on the $k$'s for which we get a formula for $L(k,\chi)$: if $\chi(-1) = 1$ then we get a formula when $k$ is even, while if $\chi(-1) = -1$ then we get a formula when $k$ is odd. This is analogous to the case of $\zeta(s)$, where we have a tidy formula for $\zeta(k)$ when $k$ is even but not odd. The zeta-function is $L(s,\chi)$ when $\chi$ is the trivial character mod 1, which is even.

A Dirichlet character $\chi$ is called *even* if $\chi(-1) = 1$ and is called *odd* if $\chi(-1) = -1$. Thus the parity of $\chi$ (even vs. odd) matches the parity of $k$ for which there are simple formulas for $L(k,\chi)$.

**Example**. If $\chi$ mod $m$ is an *odd* primitive character with $m > 1$ then

$$\begin{aligned} L(1,\chi) &= \frac{G(\chi)(2\pi i)}{2m} \sum_{j=1}^{m-1} \overline{\chi}(j)\mathbf{B}_1\left(\frac{j}{m}\right) \\ &= \frac{G(\chi)(2\pi i)}{2m} \sum_{j=1}^{m-1} \overline{\chi}(j)\left(\frac{j}{m} - \frac{1}{2}\right) \\ &= \frac{G(\chi)(2\pi i)}{2m} \left(\frac{1}{m}\sum_{j=1}^{m-1} \overline{\chi}(j)j - \frac{1}{2}\sum_{j=1}^{m-1} \overline{\chi}(j)\right) \\ &= \frac{G(\chi)(2\pi i)}{2m^2} \left(\frac{1}{m}\sum_{j=1}^{m-1} \overline{\chi}(j)j - \frac{1}{2}\sum_{j=1}^{m-1} \overline{\chi}(j)\right) \\ &= \frac{G(\chi)(2\pi i)}{2m^2} \sum_{j=1}^{m-1} \overline{\chi}(j)j. \end{aligned}$$

since $\sum_{j=1}^{m-1} \overline{\chi}(j) = 0$ on account of $\overline{\chi}$ being nontrivial.

**Remark 3.23.** . While there is not ordinarily a simple formula for $L(k,\chi)$ if $\chi$ and $k$ have opposite parity, when $\chi$ is even there *is* a concise formula for $L(1,\chi)$:

$$L(1,\chi) = -\frac{G(\chi)}{m} \sum_{j=1}^{m-1} \overline{\chi}(j)\log(\sin(\pi j/m)).$$

For example, taking $\chi(n) = (\frac{n}{5})$ (Legendre symbol mod 5) we have

$$L(1, \left(\frac{\cdot}{5}\right)) = -\frac{G(\chi)}{5}\sum_{j=1}^{4}\left(\frac{j}{5}\right)\log(\sin(\pi j/5)).$$

The Gauss sum $G(\chi) = \sum_{j \bmod 5}(\frac{j}{5})e^{2\pi i j/5}$ turns out to be $\sqrt{5}$, so

$$\begin{aligned} L(1, \left(\frac{\cdot}{5}\right)) &= -\frac{\sqrt{5}}{5}\left(\log(\sin(\frac{\pi}{5}) - \log(\sin(\frac{2\pi}{5})) - \log(\sin(\frac{3\pi}{5})) + \log(\sin(\frac{4\pi}{5}))\right) \\ &= -\frac{1}{\sqrt{5}}\log\left(\frac{\sin(\pi/5)\sin(4\pi/5)}{\sin(2\pi/5)\sin(3\pi/5)}\right). \end{aligned}$$

The product-ratio of sine values is $((3+\sqrt{5})/2)^{-1} = ((1+\sqrt{5})/2)^{-2}$, so

$$L(1, \left(\frac{\cdot}{5}\right)) = \frac{2}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right).$$

Exercises for Section 3.3

1. Let $\chi$ be a Dirichlet character mod $m$. Show $\chi$ and $\overline{\chi}$ have the same conductor. In particular, $\chi$ is primitive if and only if $\overline{\chi}$ is primitive.

2. Show the coefficient of $x^{k-1}$ in $\mathbf{B}_k(x)$ is $kB_1 = -k/2$.

3. a) Here is a characterization of the first Bernoulli polynomial. Show the only monic polynomial satisfying

$$f(X) = \sum_{j=0}^{M-1} f\left(\frac{X+j}{M}\right)$$

for all $M \in \mathbf{Z}^+$ (or even just one $M > 1$) is $X - 1/2 = \mathbf{B}_1(X)$.

b) Show, for each $M$, there is no nonzero polynomial $f(X)$ satisfying $f(X) = \sum_{j=1}^{M} f((X+j)/M)$.

c) For a nonnegative integer $k$, show

$$\mathbf{B}_k(X) = M^{k-1}\sum_{j=0}^{M-1}\mathbf{B}_k\left(\frac{X+j}{M}\right)$$

for every $M \in \mathbf{Z}^+$ by induction on $k$ using $\mathbf{B}_k'(X) = k\mathbf{B}_{k-1}(X)$.

4. Show $L(1, \chi_3) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \cdots = \frac{\pi}{3\sqrt{3}}$.

5. Show $L(3, \chi_4) = \pi^3/32$ and $L(5, \chi_4) = 5\pi^5/1536$.

# CHAPTER 4

# ANALYTIC CONTINUATION AND FUNCTIONAL EQUATION

## 4.1 Analytic Continuation of $\zeta(s)$

In his paper on the zeta-function, Riemann gave two proofs of the analytic continuation of $\zeta(s)$ to all of $\mathbf{C}$ with a simple pole at $s = 1$. His second proof is the one we will present, but since its motivation (for Riemann) depends on an observation from his first proof, we briefly discuss the first proof, omitting technical details.

Start with the formula

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nx} x^{s-1}\, dx, \tag{4.1}$$

which comes from a change of variables $x \mapsto nx$ in the definition of the Gamma function. It is possible that Riemann learned the utility of this equation from reading Dirichlet's work: in his paper on primes in arithmetic progression, Dirichlet summed (4.1) for $n \geqslant k$. (Dirichlet attributes (4.1) to Euler [**?**, p.

321].) Riemann summed (4.1) over $n \geqslant 1$ and interchanged the sum and integral:

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$

While the integral does not make sense at every $s \in \mathbf{C}$, Riemann found another expression for the integral, from which he found $\sin(\pi s)\Gamma(s)\zeta(s)$ is an entire function and

$$2\sin(\pi s)\Gamma(s)\zeta(s) = (2\pi)^s(i^{s-1} + (-i)^{s-1})\zeta(1-s).$$

After some algebra[1], Riemann found a cleaner, but equivalent, symmetry: the function $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is invariant under $s \mapsto 1-s$.

Riemann explicitly pointed to this property as his motivation for looking at a more unusual formula than (4.1):

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \int_0^\infty e^{-x}\left(\frac{x}{\pi n^2}\right)^{s/2}\frac{dx}{x} = \int_0^\infty e^{-\pi n^2 y}y^{s/2}\,\frac{dy}{y}, \tag{4.2}$$

where $s$ has real part greater than 0. (That $\int_0^\infty f(y)\,dy/y = \int_0^\infty f(cy)\,dy/y$ for $c > 0$, which is a multiplicative analogue of $\int_{-\infty}^\infty f(x+c)\,dx = \int_{-\infty}^\infty f(x)\,dx$, will be used often without comment.)

For $\mathrm{Re}(s) > 1$, sum (4.2) over all positive integers $n$. Assuming we can interchange a sum and integral, we get

$$\mathrm{Z}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \omega(y)y^{s/2}\,\frac{dy}{y}, \tag{4.3}$$

where

$$\omega(y) := \sum_{n\geqslant 1} e^{-\pi n^2 y}.$$

The series $\omega(y)$ is rapidly convergent for large $y$, and less so (but still convergent) for small $y > 0$. Integrability of $\omega(y)$ for $y$ near 0 is subtle.

To justify the interchange of sum and integral, we can use the following basic lemma, with $f_n(y) = e^{-\pi n^2 y}y^{s/2-1}$, where $\mathrm{Re}(s) > 1$.

**Lemma 4.1.** *Let $I$ be an interval in $\mathbf{R}$, $f_n(y)$ a sequence of functions that are*

[1]With $i = e^{i\pi/2}$, $i^{s-1} + (-i)^{s-1} = 2\cos(\pi(s-1)/2) = 2\sin(\pi s/2)$.

*absolutely integrable on $I$, and assume*

$$\sum_{n\geqslant 1}\int_I |f_n(y)|\,dy$$

*converges. Then the function $f(y)=\sum_{n\geqslant 1} f_n(y)$ is absolutely integrable on $I$ and*

$$\int_I f(y)\,dy = \sum_{n\geqslant 1}\int_I f_n(y)\,dy.$$

This is a straightforward application of the Dominated Convergence Theorem of Lebesgue integration. For a reader who is unfamiliar with Lebesgue integrals, it won't hurt just to take this lemma on faith. When $f_n$ and $\sum f_n$ are continuous, Lemma 4.1 is valid using Riemann integrals.

**Theorem 4.2.** *For every $c>0$, the function*

$$\int_c^\infty \omega(y)y^s\,dy = \int_c^\infty \left(\sum_{n\geqslant 1} e^{-\pi n^2 y}\right) y^s\,dy$$

*converges absolutely for all $s$, is entire, and is bounded in any left half-plane.*

This is false for $c=0$, and that is why the integral in (4.3) is subtle.

*Proof.* For $y\geqslant c>0$,

$$\omega(y)=\sum_{n\geqslant 1} e^{-\pi n^2 y}\leqslant \sum_{n\geqslant 1} e^{-\pi n y} = \frac{e^{-\pi y}}{1-e^{-\pi y}}\leqslant \frac{e^{-\pi y}}{1-e^{-\pi c}} = A_c e^{-\pi y}$$

for a constant $A_c$. For $s$ real and $y\geqslant c>0$,

$$e^{-\pi y}y^s = e^{-(\pi/2)y}e^{-(\pi/2)y}y^s\leqslant Be^{-(\pi/2)y}$$

for some constant $B$ depending on $s$ and $c$. So for every $s\in\mathbf{C}$, $\omega(y)y^{s/2-1}$ is absolutely integrable on $(c,\infty)$. The integral is entire in $s$ by Theorem 2.10. Boundedness in any left half-plane is clear. ∎

To give the integral in (4.3) a positive lower bound, Riemann appealed to a transformation property of $\omega(y)$ that he knew about from reading Jacobi:

$$\omega\left(\frac{1}{y}\right) = \frac{1}{2}(\sqrt{y}(1+2\omega(y))-1).$$

This is more cleanly stated for a series running over all integers:

$$\Theta(y) := \sum_{n\in\mathbf{Z}} e^{-\pi n^2 y} = 1 + 2\omega(y).$$

| $y$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $\Theta(y)$ | 1.003734... | 1.000161... | 1.000006... | 1.000000... |
| $\Theta(1/y)$ | 1.419495... | 1.732330... | 2.000013... | 2.236068... |

**Theorem 4.3.** *For positive $y$, $\Theta(\frac{1}{y}) = \sqrt{y}\Theta(y)$. Equivalently,*

$$\sum_{n\in\mathbf{Z}} e^{-\pi n^2/y} = \sqrt{y} \sum_{n\in\mathbf{Z}} e^{-\pi n^2 y}.$$

The proof of Theorem 4.3 will involve a detour into Fourier analysis, and we defer it to the following section.

For the algebraically inclined reader, note that Theorem 4.3 is fundamentally an analytic identity, not an algebraic one. This equality makes no sense if we think of $y$ as an indeterminate and the two sides as formal power series in $y$.

From the viewpoint of numerical analysis, Theorem 4.3 is useful for computing $\Theta(y)$ when $y$ is very small, in which case the series for $\Theta(y)$ converges slowly. At the same time $1/y$ is large, so the series for $\Theta(1/y)$ converges very rapidly. We can compute $\Theta(1/y)$ instead and then divide by $\sqrt{y}$ to obtain $\Theta(y)$.

Theorem 4.3 also tells us that for $\mathrm{Re}(s) > 1$, $\omega(y)y^{s/2-1}$ is integrable near $y = 0$ since it tells us $\omega(y) \sim 1/(2\sqrt{y})$ as $y \to 0^+$.

Returning to the zeta-function, we break up the integral in (4.3) at $y = 1$, since this is the symmetry point for $y \leftrightarrow 1/y$. Replacing $y$ with $1/y$ to convert $(0, 1]$ to $[1, \infty)$,

$$\mathrm{Z}(s) = \int_1^\infty \omega\left(\frac{1}{y}\right) y^{-s/2} \frac{dy}{y} + \int_1^\infty \omega(y) y^{s/2} \frac{dy}{y}. \tag{4.4}$$

The second integral converges for all $s$ and is bounded in any left half-plane, by Theorem 4.2. By Theorem 4.3, the first integral is

$$\begin{aligned} \int_1^\infty \frac{1}{2}(\sqrt{y}(1+2\omega(y)) - 1)y^{-s/2} \frac{dy}{y} &= \int_1^\infty \omega(y) y^{(1-s)/2} \frac{dy}{y} + \\ &\qquad \frac{1}{2}\int_1^\infty (\sqrt{y}-1) y^{-s/2} \frac{dy}{y} \\ &= \int_1^\infty \omega(y) y^{(1-s)/2} \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}. \end{aligned}$$

This integral converges for all $s$ and is bounded in any right half-plane.

We therefore have obtained an analytic formula for $\mathrm{Z}(s)$ that is valid for all $s$:

$$\begin{aligned}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_1^\infty \omega(y)(y^{s/2}+y^{(1-s)/2})\,\frac{dy}{y}-\frac{1}{s}-\frac{1}{1-s}\\ &= \int_1^\infty \frac{1}{2}(\Theta(y)-1)(y^{s/2}+y^{(1-s)/2})\,\frac{dy}{y}-\frac{1}{s}-\frac{1}{1-s}.\end{aligned}$$

The integral is bounded on the overlap of any right and left half-plane, so on any vertical strip.

Let's summarize.

**Theorem 4.4.** *The function* $\mathrm{Z}(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ *extends from* $\sigma>1$ *to a meromorphic function on* $\mathbf{C}$ *with simple poles at* $0$ *and* $1$*, having respective residues* $-1$ *and* $1$*. It satisfies the functional equation* $\mathrm{Z}(1-s)=\mathrm{Z}(s)$*, and is bounded in vertical strips except in neighborhoods of its two poles.*

We can omit each $1/2$ in the integral formula by replacing $y$ with $y^2$:

$$\mathrm{Z}(s)=\int_1^\infty(\Theta(y^2)-1)(y^s+y^{1-s})\,\frac{dy}{y}-\frac{1}{s}-\frac{1}{1-s}.$$

The function $\Theta(y^2)=\sum_{n\in\mathbf{Z}}e^{-\pi(ny)^2}$ satisfies the equation $f(1/y)=yf(y)$, which looks simpler than $f(1/y)=\sqrt{y}f(y)$.

**Corollary 4.5.** *The Riemann zeta-function has an analytic continuation to* $\mathbf{C}$ *except for a simple pole at* $s=1$ *with residue* $1$*.*

*Proof.* For each $s\in\mathbf{C}$, set $\zeta(s)=\pi^{s/2}\,\mathrm{Z}(s)/\Gamma(s/2)$. This is consistent with the previous definition of $\zeta(s)$ on $\sigma>1$. Since $\mathrm{Z}(s)$ is holomorphic except for simple poles at 0 and 1, while $\Gamma(s/2)$ has simple poles at $\{0,-2,-4,\dots\}$ and no zeros, the poles at 0 cancel and therefore $\zeta(s)$ only has a simple pole at $s=1$ with residue $\pi^{1/2}/\Gamma(1/2)=1$, as we already knew by other methods. ■

From now on, we view $\zeta(s)$ as a function on all of $\mathbf{C}$.

**Corollary 4.6 (Asymmetric Functional Equation).** *For every* $s$*,*

$$\zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

*Proof.* From $\mathrm{Z}(s) = \mathrm{Z}(1-s)$ and Theorem 2.34,

$$\zeta(1-s) = \frac{\pi^{1/2-s}\Gamma(s/2)}{\Gamma((1-s)/2)}\zeta(s) = \frac{\pi^{1/2-s}}{\Gamma((1-s)/2)}\frac{\sqrt{\pi}\Gamma(s)}{\Gamma((s+1)/2)2^{s-1}}\zeta(s).$$

This simplifies to

$$\zeta(1-s) = \frac{\pi^{1-s}\Gamma(s)}{2^{s-1}\pi}\sin\left(\frac{\pi(s+1)}{2}\right)\zeta(s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

■

The functional equation for $\mathrm{Z}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is much simpler than that for $\zeta(s)$. We call $\mathrm{Z}(s)$ the *completed zeta-function* and $\pi^{-s/2}\Gamma(s/2)$ a $\Gamma$-*factor* (the terminology is understood to include the exponential part).

Our knowledge of the zeta-function and the Gamma function implies $\mathrm{Z}(s)$ is nonvanishing for $\sigma > 1$, so also for $\sigma < 0$ by the functional equation. Therefore, since $\Gamma(s/2)$ has simple poles at the negative even integers, $\zeta(s)$ must have simple zeros at the negative even integers. These are called the *trivial* zeros of the zeta-function. The remaining zeros are in the *critical strip* $0 \leqslant \sigma \leqslant 1$, and are called *nontrivial*.

The Riemann hypothesis says that all the nontrivial zeros of the zeta-function are located on the critical line. Since the nontrivial zeros of $\zeta(s)$ are the same as the zeros of $\mathrm{Z}(s)$, with the same multiplicity, the Riemann hypothesis can be formulated as: all zeros of $\mathrm{Z}(s)$ are on the line $\sigma = 1/2$.

Exercises for Section 4.1

1. Use the asymmetric functional equation of the zeta-function and vertical estimates on the growth of the Gamma function to show that in any vertical strip $a \leqslant \sigma \leqslant b$, where $b < 0$, $1/\zeta(s) \to 0$ as $|t| \to \infty$, uniformly in $\sigma$. In particular, when $c > 0$ is not an even integer, $1/\zeta(s)$ is bounded on the line $\mathrm{Re}(s) = -c$. (We can allow $c$ to be an even integer, provided we omit a small neighborhood of the point $s = -c$.)

## 4.2 More Special Values of $\zeta(s)$

We've already computed $\zeta(s)$ at the positive even integers. With the functional equation we will compute $\zeta(s)$ at the negative integers (both even and odd), and shall find a use for the vanishing odd-indexed Bernoulli numbers. The values at $s = 0$ and $s = 1/2$ will also be computed.

**Theorem 4.7.** *For an integer* $k \geqslant 2$*,* $\zeta(1-k) = -B_k/k$*. Equivalently, for* $n \geqslant 1$*,* $\zeta(-n) = -B_{n+1}/(n+1)$*.*

*Proof.* If $k \geqslant 2$ is odd, then $B_k = 0$ and $\zeta(1-k) = 0$, so both sides are equal.

Suppose $k \geqslant 2$ is even. By the asymmetric functional equation for $\zeta(s)$,

$$\zeta(1-k) = 2(2\pi)^{-k}(k-1)!(-1)^{k/2}\zeta(k) = -\frac{B_k}{k}.$$

In the last step we used Theorem 3.2, writing $k$ in place of $2k$. ■

**Example 4.8.** $\zeta(-1) = -1/12$, $\zeta(-11) = -B_{12}/12 = 691/32760$.

The functional equation relates the zeta function at positive even integers and at negative odd integers. Values at the latter points are much cleaner. In particular, the values are rational numbers, so arithmetically interesting special values of the zeta function arise at integers where the initial Dirichlet series defining $\zeta(s)$ does not converge. (This does not mean a number like $\zeta(3)$ has no arithmetic interest. It is simply that the zeta values at negative integers are of more immediate apparent interest for number theory since they are rational.)

We neglected in Theorem 4.7 to compute $\zeta(1-k)$ when $k = 1$, i.e., $\zeta(0)$. It is reasonable to guess by the formula at negative integers that $\zeta(0)$ might be $-B_1 = 1/2$, but this is off by a sign.

**Theorem 4.9.** $\zeta(0) = -1/2$ *and* $\zeta'(0) = -(1/2)\log(2\pi) = \zeta(0)\log(2\pi)$*.*

We omit the proof.

The value of $\zeta(0)$ suggests redefining the first Bernoulli numbers to be $1/2$ rather than $-1/2$, in order that $\zeta(1-k) = -B_k/k$ for all positive integers $k$. The power series encoding such modified Bernoulli numbers is

$$1 + \frac{1}{2}x + \sum_{n \geqslant 2} \frac{B_n}{n!}x^n = \frac{x}{e^x - 1} + x = \frac{xe^x}{e^x - 1}.$$

This only changes the linear coefficient. We'll continue to use the standard definition of the Bernoulli numbers, where $B_1 = -1/2$.

Exercises for Section 4.2

1. Compute $\zeta(-n)$ for odd $n$ from 1 to 9.

2. For a positive integer $k$, $-2k$ is a simple zero of $\zeta(s)$. So $\zeta'(-2k) \neq 0$. Show
$$\zeta'(-2k) = \frac{(-1)^k (2k)!}{2(2\pi)^{2k}} \zeta(2k+1).$$
Therefore the zeta values at positive odd numbers $> 1$ appear in the first nonzero Taylor coefficient for zeta values at negative even numbers. For example, $\zeta'(-2) = -\zeta(3)/(2\pi)^2$.

## 4.3 Poisson Summation and the Theta-Function

We return to the proof of the transformation formula of Theorem 4.3,
$$\Theta\left(\frac{1}{y}\right) = \sqrt{y}\,\Theta(y), \tag{4.5}$$
where $y > 0$ and $\Theta(y) = \sum_{n\in\mathbf{Z}} e^{-\pi n^2 y} = 1 + 2\omega(y)$. This is the fundamental ingredient in the analytic continuation and functional equation of the zeta-function.

The tool that lies behind equations like (4.5) is the Poisson summation formula. To explain this, we need some details from Fourier analysis, where sums over the integers (as in the definition of $\Theta(y)$) occur repeatedly, in the following way.

The functions $e^{2\pi i n x}$ for $n \in \mathbf{Z}$ all satisfy $f(x+1) = f(x)$, *i.e.*, they have period 1. If a function $f\colon \mathbf{R} \to \mathbf{C}$ has period 1, we'd like to find conditions when $f$ admits a Fourier series representation
$$f(x) = \sum_{n\in\mathbf{Z}} c_n e^{2\pi i n x} \tag{4.6}$$
for suitable numbers $c_n$. This is a sum over all integers.

We can figure out what the numbers $c_n$ should be. First, for a nonzero integer $k$
$$\int_0^1 e^{2\pi i k x}\, dx = \left.\frac{e^{2\pi i k x}}{2\pi i k}\right|_0^1 = 0,$$

while if $k = 0$ the integral is 1. So

$$\int_0^1 f(x)e^{-2\pi imx}\,dx = \sum_{n\in\mathbf{Z}} \int_0^1 c_n e^{2\pi i(n-m)x}\,dx = c_m.$$

Note the minus sign in the exponent of the integrand on the left. To extract the $m$th coefficient in (4.6), we want to integrate $f(x)$ against $e^{-2\pi imx}$ to kill off every term except the $m$th.

(There is no intrinsic difference between $i$ and $-i$, so the role of $-2\pi i$ in the integral formula for $c_m$ could be played by $2\pi i$ if we pay attention to how we write Fourier series. Namely, upon fixing either choice of $\sqrt{-1}$, we write Fourier series in the form $f(x)\sum c_n e^{2\pi\sqrt{-1}nx}$, and then the formula for $c_m$ involves an integral of $f(x)e^{2\pi(-\sqrt{-1})mx}$.)

Here is the basic theorem we need on (pointwise) convergence of Fourier series.

**Theorem 4.10.** *Let $f\colon \mathbf{R} \to \mathbf{C}$ have period 1 and be continuous. Define the Fourier coefficients $c_n$ by the formula*

$$c_n = \int_0^1 f(x)e^{-2\pi inx}\,dx. \tag{4.7}$$

*a) If $\sum_{n\in\mathbf{Z}} |c_n|$ converges, then*

$$f(x) = \sum_{n\in\mathbf{Z}} c_n e^{2\pi inx}$$

*for all $x$.*

*b) If for an integer $k \geqslant 2$, $c_n = O(1/n^k)$ as $n \to \infty$, then $f$ is continuously differentiable $k-2$ times and the corresponding termwise differentiations of the Fourier series of $f$ converge to the corresponding higher derivatives of $f$.*

*c) If the real and imaginary parts of $f$ are continuously differentiable, then for all $x$*

$$f(x) = c_0 + \sum_{n\geqslant 1}(c_n e^{2\pi inx} + c_{-n}e^{-2\pi inx}) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi inx}.$$

*d) If $f$ is piecewise continuously differentiable on $[0,1]$ with a bounded derivative, except for finitely many discontinuities, then the Fourier expansion in part c) holds at all points where $f$ is continuous. At other points the Fourier series* *converges to the average of the right and left hand limits:*

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi inx} = \frac{1}{2}\left(f(x+) + f(x-)\right),$$

*where* $f(x+) := \lim_{u\to x+} f(u)$*, and* $f(x-) := \lim_{u\to x-} f(u)$*.*

*Proof.* We refer the reader to Körner's book [?] on Fourier analysis. Part a) is Theorem 9.1 in [?] (where the domain and range should be interchanged). Part b) is Theorem 9.4. Part c) is Theorem 15.4. Part d) is Theorem 16.4. ∎

Almost all the functions we use will be infinitely differentiable, so the reader shouldn't worry about Fourier series of nonsmooth functions, whose pointwise behavior can be rather erratic. In light of the formula in part d) for the value of a Fourier series at a point of discontinuity, the condition $f(x) = (1/2)(f(x+) + f(x-))$ is a natural weakening of the continuity condition. Functions satisfying this property are called *normalized*.

The function $\Theta(y)$ is nonperiodic. How do we introduce periodic functions (and then their Fourier series) to prove something about a nonperiodic function? By the process of *periodization*. Starting with a nonperiodic function $h$, we can create a function with period 1 by summing integer translates:

$$H(x) = \sum_{n\in\mathbf{Z}} h(x+n).$$

Clearly $H(x+1) = H(x)$. This gives us a means of associating to a function $h(x)$ on $\mathbf{R}$ a function $H(x)$ with period 1, assuming of course that the sums converge.

If we could write $H(x) = \sum_{n\in\mathbf{Z}} c_n e^{2\pi inx}$ for some numbers $c_n$, let's determine what the Fourier coefficients of $H$ are, in terms of the original function $h$:

$$c_n = \int_0^1 H(x)e^{-2\pi inx}\,dx = \sum_{m\in\mathbf{Z}} \int_m^{m+1} h(x)e^{-2\pi inx}\,dx = \int_{-\infty}^{\infty} h(x)e^{-2\pi inx}\,dx.$$

This last expression is analogous to the formula (4.7) for the coefficients in a Fourier series, but it is an integral over $\mathbf{R}$ rather than over $[0,1]$. (While $H(x)$

has period 1, $h(x)$ does not.) Taking $x = 0$,

$$\sum_{n\in\mathbf{Z}} h(n) = H(0) = \sum_{n\in\mathbf{Z}} c_n = \sum_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} h(x)e^{-2\pi inx}\,dx. \tag{4.8}$$

**Definition 4.11.** The *Fourier transform* of an absolutely integrable function $f\colon \mathbf{R}\to\mathbf{C}$ is the function $\mathcal{F}f\colon \mathbf{R}\to\mathbf{C}$ given by the formula

$$(\mathcal{F}f)(y) = \widehat{f}(y) := \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx.$$

For example, $\widehat{f}(n) = \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx$, so $\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\,dx$. When it is defined, the Fourier transform is $\mathbf{C}$-linear: $(f+g)\widehat{\ } = \widehat{f}+\widehat{g}$ and $\widehat{cf} = c\widehat{f}$ for $c \in \mathbf{C}$. We may write $\mathcal{F}f$ rather than $\widehat{f}$ when we want to emphasize the role of the Fourier transform as an operator on functions.

In the Fourier transform notation, (4.8) takes the form

$$\sum_{n\in\mathbf{Z}} h(n) = \sum_{n\in\mathbf{Z}} \widehat{h}(n). \tag{4.9}$$

This is called the *Poisson summation formula*. For a choice of $h(x)$ to be made below, it will yield the transformation law for $\Theta(y)$.

**Theorem 4.12.** *Assume $h$ is continuous and $|h(x)| \leqslant A/|x|^c$ for large $x$, where $A$ and $c$ are positive constants, $c > 1$. Then $\widehat{h}(y)$ is defined for all $y$, and if also $\sum_{n\in\mathbf{Z}} |\widehat{h}(n)|$ converges then the Poisson summation formula for $h$ is valid.*

An equivalent decay bound on $h(x)$ that can apply to all $x$ is $|h(x)| \leqslant A/(1+|x|)^c$, where $A, c$ may be different than in the theorem.

*Proof.* The assumptions on $h$ imply $H(x) = \sum_{n\in\mathbf{Z}} h(x+n)$ converges absolutely for each $x$ and converges uniformly in $x$ on each compact subset of $\mathbf{R}$, so $H$ is continuous. Since the $n$th Fourier coefficient of $H$ is $\widehat{h}(n)$, the additional assumption on convergence of $\sum_{n\in\mathbf{Z}} |\widehat{h}(n)|$ (which, for instance, is assured if $|\widehat{h}(y)|$ has the same type of decay as $h$, like some power $1/|y|^{c'}$ where $c' > 1$) justifies Poisson summation by part a) of Theorem 4.10. ∎

For a more flexible Poisson summation, replace $h(x)$ by $g(x) = h(ax+b)$ for a nonzero constant $a$. The Fourier transform changes by an exponential factor:

$$g(x) = h(ax+b) \Longrightarrow \widehat{g}(y) = \int_{-\infty}^{\infty} h(ax+b)e^{-2\pi ixy}\,dx = \frac{e^{2\pi iby/a}}{|a|}\widehat{h}\left(\frac{y}{a}\right). \tag{4.10}$$

One particularly useful case is the effect of the Fourier transform under "interior scaling": $h(ax)$ has Fourier transform $(1/|a|)\widehat{h}(y/a)$.

Poisson summation for $g(x) = h(ax+b)$ takes the form

$$\sum_{n\in\mathbf{Z}} h(an+b) = \frac{1}{|a|}\sum_{n\in\mathbf{Z}} e^{2\pi ibn/a}\widehat{h}\left(\frac{n}{a}\right). \tag{4.11}$$

In particular, setting $b = 0$ and $a = 1$ gives the respective equations

$$\sum_{n\in\mathbf{Z}} h(an) = \frac{1}{|a|}\sum_{n\in\mathbf{Z}} \widehat{h}\left(\frac{n}{a}\right), \quad \sum_{n\in\mathbf{Z}} h(n+b) = \sum_{n\in\mathbf{Z}} e^{2\pi ibn}\widehat{h}(n).$$

It is natural to ask how the Fourier transform interacts with other basic operations in analysis. We've already seen that it is linear. Let's consider differentiation. Our functions $f$ are taking complex values, so differentiation can be regarded as applied to the real and imaginary parts of $f$. Differentiating under the integral sign suggests that

$$(\mathcal{F}f)'(y) = (\widehat{f})'(y) = -2\pi i\int_{-\infty}^{\infty}(xf(x))e^{-2\pi ixy}\,dx = -2\pi i\widehat{(xf)}(y),$$

and

$$\mathcal{F}(f')(y) = \widehat{f'}(y) = \int_{-\infty}^{\infty} f'(x)e^{-2\pi ixy}\,dx = 2\pi iy\widehat{f}(y)$$

by an integration by parts (assuming $f$ vanishes at $\pm\infty$). Letting $(Mf)(x) = xf(x)$ be the operation of multiplication by the variable and and $(Df)(x) = f'(x)$ be differentiation, the above suggestive equations take the form

$$D\widehat{f} = -2\pi i\widehat{Mf}, \quad \widehat{Df} = 2\pi iM\widehat{f}. \tag{4.12}$$

So if there is a vector space of functions on $\mathbf{R}$ that is closed under the Fourier transform and differentation, then it should be closed under multiplication by $x$, and thus under multiplication by every polynomial. A function has to decay fairly rapidly for large $|x|$ if it (and all its derivatives) will still remain integrable over $\mathbf{R}$ after multiplication by every polynomial. We are led to the following space.

**Definition 4.13.** The *Schwartz space* on $\mathbf{R}$, $\mathcal{S}(\mathbf{R})$, is the set of infinitely differentiable functions $f\colon \mathbf{R} \to \mathbf{C}$ where $x^m f^{(n)}(x)$ is bounded for all integers $m, n \geqslant 0$.

We will say a function $h$ is rapidly decreasing at infinity if $x^m h(x) \to 0$ as $|x| \to \infty$ for every positive integer $m$. Functions in the Schwartz space have all their derivatives rapidly decreasing at infinity. Note $1/(1+x^2)$ and all of its derivatives tend to 0 as $|x| \to \infty$, but they are not rapidly decreasing according to the definition of that term (multiply by $x^3$). So $1/(1+x^2)$ is not in $\mathcal{S}(\mathbf{R})$. The most basic example of a nonzero function in the Schwartz space is $e^{-x^2}$, and then $P(x)e^{-x^2}$ for every polynomial $P(x)$. Another example is an infinitely differentiable function that vanishes outside a bounded interval. The only polynomial (or rational function) in the Schwartz space is 0. If $f(x) \in \mathcal{S}(\mathbf{R})$ then $f(ax+b) \in \mathcal{S}(\mathbf{R})$ for constants $a, b$ with $a \neq 0$.

The Fourier transform of a function in the Schwartz space is bounded, since $|\widehat{f}(y)| \leqslant \int_{-\infty}^{\infty} |f(x)|\,dx$, and we can differentiate $\widehat{f}(y)$ under the integral sign as often as desired. This leads to

$$\begin{aligned} f \in \mathcal{S}(\mathbf{R}) \quad &\Longrightarrow \quad D^n(\widehat{f}) = (-2\pi i)^n \widehat{x^n f} \\ &\Longrightarrow \quad y^m D^n(\widehat{f}) = (-2\pi i)^n y^m \widehat{x^n f} = (-1)^n (2\pi i)^{n-m} \mathcal{F}(D^m(x^n f)). \end{aligned}$$

So if $f \in \mathcal{S}(\mathbf{R})$ then $y^m D^n(\widehat{f})$ is bounded for all $m, n$, so $\widehat{f} \in \mathcal{S}(\mathbf{R})$.

All the operations we have performed with the Fourier transform are valid for functions in the Schwartz space, since convergence issues that arise in proofs are easy to handle on account of the rapid decay. In particular, the Poisson summation formula is true for functions in the Schwartz space, as Theorem 4.12 easily applies. (For an example where Theorem 4.12 applies outside the Schwartz space, use $f(x) = e^{-a|x|}$ for $\mathrm{Re}(a) > 0$. Then $\widehat{f}(y) = 2a/(a^2 + 4\pi^2 y^2)$.)

The following calculation will play a special role in our treatment of $\zeta(s)$ and $L(s, \chi)$. It also gives an application of the formula (4.12) relating the Fourier transform and differentiation.

**Theorem 4.14.** *The function $f(x) = e^{-\pi x^2}$ equals its own Fourier transform: $\widehat{f}(y) = e^{-\pi y^2}$.*

*Proof.* We give two proofs.

For the first proof, note $e^{-\pi x^2}$ is a solution to the differential equation $f'(x) + 2\pi x f(x) = 0$, and all other solutions are constant multiples of $e^{-\pi x^2}$. Apply the Fourier transform to this differential equation, using (4.12), to get

$$2\pi i y \widehat{f}(y) + i(\widehat{f})'(y) = 0.$$

Cancel the $i$, so $\widehat{f}$ satisfies the same differential equation as $f$. Thus, when $f(x) = e^{-\pi x^2}$, $\widehat{f}(x) = af(x)$ for some constant $a$. Set $x = 0$ to get $a = \widehat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$. Saying $a = 1$ is equivalent to

$$\int_{-\infty}^{\infty} e^{-(1/2)u^2}\,du = \sqrt{2\pi},$$

which was proved already in Section 2.3. This concludes the first proof.

The equations $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(1/2)u^2}\,du = 1$ and $\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = 1$ are the same, but the second one will be more convenient for our work and the reader should memorize it.

The second proof of the theorem consists in rewriting the equation $\widehat{f}(y) = e^{-\pi y^2}$ in the form

$$\int_{-\infty}^{\infty} e^{-\pi(x-iy)^2}\,dx = \int_{\{x-iy:x\in\mathbf{R}\}} e^{-\pi z^2}\,dz = 1, \tag{4.13}$$

which indicates the left side should be independent of $y$.

The fact that at $y = 0$ the integral in (4.13) equals 1 has already been treated. For $y \neq 0$, take a rectangular contour with vertices at $-R, R, R - iy$, and $-R-iy$. The integral of $e^{-\pi z^2}$ around this contour is 0 by Cauchy's theorem. The integral along the vertical edges tends to 0 as $R \to \infty$, so the integrals along the two horizontal edges, when oriented in the same direction, coincide in the limit. So the left side of (4.13) is independent of $y$. ∎

The fact that $e^{-\pi x^2}$ equals its own Fourier transform depends on the way we defined the Fourier transform. For instance, in physics the Fourier transform is usually defined using the function $e^{-ixy}$, without the $2\pi$ factor in the exponent, but instead the $2\pi$ (to some power) winds up as a factor outside the integral, Fourier series coefficients are computed as integrals over $[0, 2\pi]$, not $[0, 1]$, and the statement of Poisson summation has a $2\pi$ appearing on one side. In such a language, $e^{-(1/2)x^2}$ equals its own Fourier transform. For a table of the possible constants that are used in the different definitions of the Fourier transform, see [**?**, Appendix F].

The next result proves Theorem 4.3, and so fills in the gap in our proof of the analytic continuation and functional equation for the Riemann zeta-function.

**Theorem 4.15 (Poisson, 1823).** *For every positive real number $r$,*

$$\sum_{n\in\mathbf{Z}} e^{-\pi n^2/r^2} = r\sum_{n\in\mathbf{Z}} e^{-\pi n^2 r^2}.$$

*For $y>0$, writing $y=r^2$, $\Theta(1/y)=\sqrt{y}\Theta(y)$.*

*Proof.* Using the machinery we have already developed, just apply the Poisson summation formula in the form (4.11) to $h(x)=e^{-\pi x^2}$, $a=r$, $b=0$. We're done. ∎

When a function equals its Fourier transform, it is called *self-dual*. For example, $e^{-\pi x^2}$ is self-dual. This is not the only function (up to scaling) that is self-dual. To obtain more examples, we establish in the next theorem a property that allows us to recover a function in $\mathcal{S}(\mathbf{R})$ from its Fourier transform. We will abbreviate $\int_{-\infty}^{\infty}$ to $\int_{\mathbf{R}}$.

**Theorem 4.16 (Fourier Inversion).** *For any $f\in\mathcal{S}(\mathbf{R})$, $(\mathcal{F}^2 f)(y)=f(-y)$. Equivalently,*

$$f(y)=\int_{\mathbf{R}}\widehat{f}(x)e^{2\pi ixy}\,dx.$$

*Proof.* A naive computation of the iterated Fourier transform yields

$$\begin{aligned}(\mathcal{F}^2 f)(y) &= \int_{\mathbf{R}}(\mathcal{F}f)(x)e^{-2\pi ixy}\,dx\\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} f(u)e^{-2\pi ix(u+y)}\,du\,dx\\ &= \int_{\mathbf{R}}\int_{\mathbf{R}} f(u-y)e^{-2\pi ixu}\,du\,dx.\end{aligned}$$

At this point we're stuck. We can't interchange the integrals, since $e^{-2\pi ixu}$ is not integrable (with respect to $x$) over $\mathbf{R}$ in the usual sense. There is a technical trick to circumvent this problem, as follows.

We introduce a factor into the integral that decays quickly enough at $\pm\infty$ so the calculation can continue, but also is essentially 1 on a large enough interval so that the integral is close to $(\mathcal{F}^2 f)(y)$.

Let $h(x)=e^{-\pi x^2}$, so $\widehat{h}=h$. For $\varepsilon>0$, let $h_\varepsilon(x)=h(\varepsilon x)$. This function is in the Schwartz space, but for very small $\varepsilon$ it is essentially equal to $h(0)=1$ unless $x$ is large.

Consider

$$\begin{aligned}\int_{\mathbf{R}} \widehat{f}(x)e^{-2\pi ixy}h(\varepsilon x)\,dx &= \int_{\mathbf{R}}\int_{\mathbf{R}} f(u)e^{-2\pi ix(u+y)}h(\varepsilon x)\,du\,dx \\ &= \int_{\mathbf{R}} f(u)\widehat{h_\varepsilon}(u+y)\,du.\end{aligned}$$

Since $\widehat{h_\varepsilon}(y) = (1/\varepsilon)\widehat{h}(y/\varepsilon) = (1/\varepsilon)h(y/\varepsilon)$,

$$\int_{\mathbf{R}} \widehat{f}(x)e^{-2\pi ixy}h(\varepsilon x)\,dx = \int_{\mathbf{R}} f(u)\frac{1}{\varepsilon}h\left(\frac{u+y}{\varepsilon}\right)\,du = \int_{\mathbf{R}} f(\varepsilon v - y)h(v)\,dv.$$

The left and right sides are both continuous in $\varepsilon$. (Check this.) Then letting $\varepsilon \to 0^+$, we get

$$(\mathcal{F}^2 f)(y) = \int_{\mathbf{R}} \widehat{f}(x)e^{-2\pi ixy}\,dx = \int_{\mathbf{R}} f(-y)h(v)\,dv = f(-y).$$

■

By Fourier inversion, for every $f \in \mathcal{S}(\mathbf{R})$ the function

$$f(x) + (\mathcal{F}f)(x) + (\mathcal{F}^2 f)(x) + (\mathcal{F}^3 f)(x) = f(x) + \widehat{f}(x) + f(-x) + \widehat{f}(-x)$$

is self-dual. Taking $f$ to be even, $f + \widehat{f}$ is self-dual. So in principle there are many self-dual functions in $\mathcal{S}(\mathbf{R})$.

Exercises for Section 4.3

1. Prove directly that $P(x)e^{-x^2}$ is in the Schwartz space for every polynomial $P(x)$.

2. For a real quadratic polynomial $q(x)$ with positive leading coefficient, show $e^{-q(x)} \in \mathcal{S}(\mathbf{R})$.

3. a) For $a > 0$, compute the Fourier transform of $e^{-ax^2}$.

   b) For $a > 0$ and $w \in \mathbf{C}$, show $e^{-a(x+w)^2}$ is in $\mathcal{S}(\mathbf{R})$, and compute its Fourier transform.

4. Let $f \in \mathcal{S}(\mathbf{R})$.

   a) Show $g(x) = f(-x)$ has Fourier transform $\widehat{g}(y) = \widehat{f}(-y)$. In other words, negating the variable and taking the Fourier transform are commuting operations.

b) Show $f$ is real-valued and even if and only if $\widehat{f}$ is real-valued and even.

5. a) Let $g(x) = e^{2\pi i r x} f(x+c)$ for $f$ in the Schwartz space and $r, c$ real constants. Show $\widehat{g}(y) = e^{2\pi i c(y-r)} \widehat{f}(y-r)$.

   b) Fix $r > 0$. Show there is a nonzero $f \in \mathcal{S}(\mathbf{R})$ such that $f$ vanishes on $r\mathbf{Z}$ and $\widehat{f}$ vanishes on $(1/r)\mathbf{Z}$.

   c) Give infinitely many linearly independent solutions to part b).

   d) Show $(\sin x)/x$ and all of its derivatives are bounded functions on $\mathbf{R}$ that do not belong to $\mathcal{S}(\mathbf{R})$.

6. a) For real $y$, show

$$\int_0^\infty e^{-\pi(x+iy)^2}\,dx = \frac{1}{2} - i\int_0^y e^{\pi u^2}\,du.$$

   b) Let $f(x) = (\text{sign } x)e^{-\pi x^2}$, with $f(0) = 0$. (Here sign $x = 1$ if $x > 0$ and sign $x = -1$ if $x < 0$.) This function has a discontinuity at 0 (so it is not in the Schwartz space), but otherwise is smooth and decays to 0 very rapidly. Show

$$\widehat{f}(y) = -ie^{-\pi y^2}\int_{-y}^{y} e^{\pi u^2}\,du = -2ie^{-\pi y^2}\int_0^y e^{\pi u^2}\,du$$

   c) As $|y| \to \infty$, check that $\widehat{f}(y) \to 0$ and $y\widehat{f}(y)$ has a finite nonzero limit (so $|y^{1+\varepsilon}\widehat{f}(y)| \to \infty$ for every $\varepsilon > 0$). Therefore $\widehat{f}$ doesn't decay to 0 that quickly.

7. a) Show the hyperbolic secant $\operatorname{sech} x = 2/(e^x + e^{-x})$ is in the Schwartz space.

   b) Show $\operatorname{sech}(\pi x)$ is self-dual, *i.e.*,

$$\int_{-\infty}^\infty \frac{2}{e^{\pi x} + e^{-\pi x}} e^{-2\pi i x y}\,dx = \frac{2}{e^{\pi y} + e^{-\pi y}},$$

   by applying the residue theorem to the contour with vertices $-R, R, R+i, -R+i$, and let $R \to \infty$. Also give a heuristic proof treating $\mathbf{R}$ as a contour "surrounding" the upper half-plane $\mathfrak{H}$. Does it matter whether $\mathbf{R}$ is viewed as a contour surrounding the upper half-plane or the lower half-plane?