Lecture 4: Drinfeld modules!

Carlitz module: The Drinfeld module of rank 1

# fld	function fld
$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$	$e_C(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{D_n}$
	$D_n = \prod_{\substack{a \in \mathbb{F}_q[T] \\ a \text{ monic} \\ \deg a = n}} a$

$\exp(z)$	$e_C(z)$
• holomorphic	• entire (ex Lec 2)
• $\exp: \mathbb{C} \to \mathbb{C}^\times$ surjective	• surjective $e_C : C_\infty \to C_\infty$
• $\ker \exp = \{z : \exp(z) = 1\} = \underbrace{2\pi i}_{\text{transcendental}} \mathbb{Z}$	• $\ker e_C(z) = \{z : e_C(z) = 0\} = \underbrace{\tilde{\pi}}_{\text{transcendental}} \mathbb{F}_q[T] = \Lambda$
	$e_C(z) = z \prod_{\lambda \in \Lambda'} \left(1 - \frac{z}{\lambda}\right)$

- $\exp(nz) = (\exp(z))^n$

  $n \in \mathbb{Z}$

this assigns

$$n \rightsquigarrow z^n$$

- torsion points

  $z^n = 1$

  generate abelian extensions of $\mathbb{Q}$

- $\forall a \in \mathbb{F}_q[T]\ \exists C_a \in \mathbb{F}_q[T]\{\tau\}$

s.t.

$$e_C(az) = C_a(e_C(z))$$

$$a \rightsquigarrow C_a(z)$$

- torsion points

  $C_a(z) = 0$

  generate most abelian extensions of $\mathbb{F}_q(T)$

Proposition $\qquad \mathbb{F}_q[T]$

$$e_C(Tz) = Te_C(z) + (e_C(z))^q$$

proof: $$e_C(Tz) = \sum_{n=0}^{\infty} \frac{(Tz)^{q^n}}{D_n}$$

$$e_C(Tz) - Te_C(z) = \sum_{n=0}^{\infty} \frac{T^{q^n} z^{q^n} - Tz^{q^n}}{D_n}$$

$$= \sum_{n=1}^{\infty} \frac{(T^{q^n} - T)\, z^{q^n}}{D_n}$$

Aside

$$D_n = (T^{q^n} - T)\, D_{n-1}^q$$

$$e_C(Tz) - Te_C(z) = \sum_{n=1}^{\infty} \frac{z^{q^n}}{D_{n-1}^{q}}$$

$$= \left(\sum_{n=1}^{\infty} \frac{z^{q^{n-1}}}{D_{n-1}} \right)^{q}$$

$$= (e_C(z))^q$$

$\square$

Corollary: For all $a \in \mathbb{F}_q[T]$ $\exists! C_a \in \mathbb{F}_q[T]\{\tau\}$

s.t. $e_C(az) = C_a(e_C(z))$

$a = T^2 + 2$

$$e_c((T^2+2)z) = e_c(\overbrace{T^2 z}^{T(Tz)}) + 2e_c(z) \qquad (\mathbb{F}_q\text{-lin})$$

$$= Te_c(Tz) + e_c(Tz)^q + 2e_c(z)$$

$$= T^2 e_c(z) + Te_c(z)^q + T^q e_c(z)^q + e_c(z)^{q^2}$$

$$+ 2e_c(z)$$

$$= (T^2+2)e_c(z) + (T^q+T)e_c(z)^q + e_c(z)^{q^2}$$

$$= C_{T^2+2}(e_c(z))$$

$$C_{T^2+2} = (T^2+2)\tau^0 + (T^q+T)\tau + \tau^2$$

$\{z : z^n = 1\} \cong \mathbb{Z}/n$

$\cong \mu_n$

$\{z : C_a(z) = 0\} \cong \mathbb{F}_q[T]/(a)$

Theorem (Hayes)

The extensions of $\mathbb{F}_q(T)$ generated by these torsion points + constant field extensions ($\mathbb{F}_{q^2}(T)$ ...) contain all abelian extensions of $\mathbb{F}_q(T)$ that are not wildly ramified at ∞.

If I want those, start over with $\frac{1}{T}$ instead of T

Drinfeld modules of rank 2

Let K be an $\mathbb{F}_q[T]$-field : $\exists\ \iota : \mathbb{F}_q[T] \xrightarrow{\text{ring hom}} K$

Def

A homomorphism of $\mathbb{F}_q$-algebras

$$\varphi : \mathbb{F}_q[T] \to K\{\tau\} = \mathrm{End}_{\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a/K})$$

is a DM / K iff

- $\varphi_a'(z) = \iota(a) \Leftrightarrow \varphi_a(z) = \iota(a)z + \dots\dots$
- for some a, $\varphi_a \neq a\tau^0$

$K = \mathbb{F}_q(T)$ $\quad \varphi_a(z) \in \mathbb{F}_q(T)[z]$

$z, z^q \ldots$

This DM is of rank 2 iff

$$\boxed{\varphi_T(z) = T\tau^0 + g\tau + \Delta\tau^2 \quad \Delta \neq 0}$$

$(\ C_T(z) = T\tau^0 + \tau^1)$

$\varphi_{T^2}(z)$

Def: An isogeny $P: \varphi \to \psi$ is a non zero $P \in \bar{K}\{\tau\}$

s.t. $\psi_a \cdot P = P \cdot \varphi_a \quad \forall a \in \mathbb{F}_q[T]$

composition

An isomorphism is an invertible isogeny

i.e. $P = c \in \bar{K}$

When are ψ & φ isomorphic?

$\exists\, c \in \bar{K}$

$$\psi_T \cdot (cz) = (cz) \cdot \varphi_T$$

$$\| \qquad //$$

$$(T\tau^0 + g_2\tau + \Delta_2\tau^2)\cdot(c\tau^0) = Tc\tau^0 + g_2 c^q \tau + \Delta_2 c^{q^2}\tau^2$$

$$\|$$

$$(c\tau^0)(T\tau^0 + g_1\tau + \Delta_1\tau^2) = Tc\tau^0 + g_1 c\tau + \Delta_1 c\tau^2$$

$$g_2 c^q = g_1 c \qquad \Delta_2 c^{q^2} = \Delta_1 c$$

$$g_1 = c^{q-1} g_2 \qquad \Delta_1 = c^{q^2-1}\Delta_2$$

$$j = \frac{g^{q+1}}{\Delta}$$

$\longrightarrow \infty = j$

j-line

$\Delta = 0$

Isogenies of Drinfeld modules over C_∞

$$\begin{tikzcd}
C_\infty \arrow[r, "e_{\Lambda_1}"] \arrow[d, "\alpha"'] & C_\infty \arrow[d, "P_\alpha"] \\
C_\infty \arrow[r, "e_{\Lambda_2}"'] & C_\infty
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_1 \arrow[r] \arrow[d] \arrow[dd, "az"'] & C_\infty \arrow[r, "e_{\Lambda_1}"] \arrow[d] \arrow[dd, "az"'] & C_\infty \arrow[r] \arrow[d, "\varphi_a"] \arrow[dd, "P_\alpha"'] & 0 \\
0 \arrow[r, "\alpha"] & \Lambda_1 \arrow[r, "\alpha"] \arrow[dd] & C_\infty \arrow[r, "e_{\Lambda_1}"] \arrow[dd] & C_\infty \arrow[r] \arrow[dd, "P_\alpha"] & 0 \\
0 \arrow[r] & \Lambda_2 \arrow[r] \arrow[d, "az"'] & C_\infty \arrow[r, "e_{\Lambda_2}"] \arrow[d, "az"'] & C_\infty \arrow[r] \arrow[d, "\psi_a"] & 0 \\
0 \arrow[r] & \Lambda_2 \arrow[r] & C_\infty \arrow[r, "e_{\Lambda_2}"'] & C_\infty \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
C_\infty \arrow[d, "\psi_a"'] & C_\infty \arrow[l, "P_\alpha"'] \arrow[d, "\varphi_a"] \\
C_\infty & C_\infty \arrow[l, "P_\alpha"]
\end{tikzcd}$$