Lecture 2

Previously

- field $\mathbb{F}_q(T)$, Ring $\mathbb{F}_q[T]$, $q=p^r$
- $v(x) = v\left(\frac{a}{b}\right) = \deg b - \deg a$
- $|x| = \left|\frac{a}{b}\right| = q^{\deg a - \deg b} = q^{-v(x)}$
- properties of non-Archimedean fields
  - ultrametric inequality
  - Δ is isosceles
  - balls are disjoint or contained in each other

True facts

- if K is of char p then every $|\cdot|$ is non-Archimedean

- every Archimedean $|\cdot|$ comes from

  $\iota : K \hookrightarrow \mathbb{C}$ $\qquad |x| = |\iota(x)|_\infty^\alpha \qquad \alpha = 1, 2$

Why?

Analogy	# fields	function fields
	$\mathbb{Z}$	$\mathbb{F}_q[T]$
	$\mathbb{Q}$	$\mathbb{F}_q(T)$

- it works, many theorems have analogues on either side
- both $\mathbb{Z}$, $\mathbb{F}_q[T]$ are PIDs

  "base rings" Dedekind domains
- completions of $\mathbb{Q}$ & $\mathbb{F}_q(T)$ are all locally compact fields
- both $\mathbb{Z}$, $\mathbb{F}_q[T]$ have finite unit groups
- both $\mathbb{Q}$, $\mathbb{F}_q(T)$ are "global fields"

FOR US: class field theory, elliptic curves

First, finish completions & entire functions

Definitions

Let K be a fld, $|\cdot|$ abs value on K

① $\{x_n\}_{n=0}^{\infty}$ $x_n \in K$ this sequence is Cauchy

if $\forall \varepsilon > 0$ $\exists N$ s.t. $|x_n - x_m| < \varepsilon$ whenever

$n, m > N$

② K is complete ~~iff~~ wrt $|\cdot|$ if every Cauchy sequence converges to an element of K

③ $S \subset K$ is dense if every open ball around every element of K contains an element of S

i.e. $\forall x \in K \ \forall \varepsilon > 0 \quad B(x, \varepsilon) \cap S \neq \emptyset$

My problem: $\mathbb{F}_q(T)$ is not complete

Consider $$x_n = \sum_{i=-n}^{0} T^i = T^{-n} + T^{-n+1} + \dots + 1$$

$$= \frac{T^n + T^{n-1} + \dots + 1}{T^n} \in \mathbb{F}_q(T)$$

$\{x_n\}_{n=0}^{\infty}$ is Cauchy

Fix $\varepsilon > 0$, let N be s.t. $q^{-N} < \varepsilon$

Then if $n \geq m \geq N$

$$|x_n - x_m| = |T^{-n} + T^{-n+1} + \dots + T^{-m-1}|$$

$$= \max(|T^{-n}|, |T^{-n+1}|, \dots |T^{-m-1}|)$$

$$= \max(q^{-m}, q^{-n+1} \dots q^{-m-1})$$

$$= q^{-m-1} < q^{-N} < \varepsilon$$

This "should" converge to $x_\infty = \sum_{i=-\infty}^{0} T^i \in \mathbb{F}_q(T)$

$$= \frac{\frac{1}{T}}{\frac{1}{T} - 1}$$

Instead $x_n = \sum_{i=-n}^{0} T^{i^2} \longrightarrow x_\infty = \sum_{i=-\infty}^{0} T^{i^2} \notin \mathbb{F}_q(T)$

↑ Cauchy

Want a place where every Cauchy sequences converges, this is called the completion of K

"Definition" $\hat{K}$ the completion of K wrt $|\cdot|$ is $K \cup \{$the limit of every Cauchy sequence$\}$

Look up the real definition

FOR US $\widehat{\mathbb{F}_q(T)} = \mathbb{F}_q((\frac{1}{T})) \ni \sum_{n=-\infty}^{N} a_n T^n$

$\pi \simeq 3.14159 \ldots$

Proposition

Let $\hat{K}$ be a complete fld wrt a non-Arch $|\cdot|$.

① $\{x_n\}$ is Cauchy iff $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$

② $\sum_{n=0}^{\infty} x_n$ converges iff $\lim_{n\to\infty} x_n = 0$

$\mathbb{Z}$ $\qquad$ $\mathbb{F}_q[T]$

$\mathbb{Q}$ $\qquad$ $\mathbb{F}_q(T)$

$\mathbb{R}$ $\qquad$ $\mathbb{F}_q((\frac{1}{T})) = K_\infty$ – complete

complete & alg closed $\to$ $\mathbb{C}$ $\qquad$ $\left\{ \begin{array}{l} \overline{\mathbb{F}_q((\frac{1}{T}))} \text{ – alg closed but not complete} \\ \widehat{\overline{\mathbb{F}_q((\frac{1}{T}))}} \text{ – complete \& alg closed} \end{array} \right.$

$\widehat{\overline{\mathbb{F}_q((\frac{1}{T}))}} = C_\infty$

$[\mathbb{C} : \mathbb{R}] = 2$

$[C_\infty : K_\infty] = \infty$

Definition / Proposition

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \quad a_n \in C_\infty$. We say

f is <u>entire</u> if it converges for all $x \in C_\infty$

This happens iff $\lim_{n\to\infty} \frac{v(a_n)}{n} = \infty$

Next up: char p stuff

Additive & $\mathbb{F}_q$-linear polynomials.

Def: Let K be a fld, $P(x) \in K[x]$ then

P is additive if

$$P(x+y) = P(x) + P(y) \quad \forall\, x, y \in K$$

$K = \mathbb{Q}$ only additive polys $P(x) = cx$

But in char p, we have $(x+y)^p = x^p + y^p$

Also: $P(x) \in \mathbb{F}_3[x]$ check that

$$P(x) = x^6 + x^4 + x^2 + x \quad \text{is additive}$$

Def: Let K be a fld & $\mathbb{F}_q \subset K$

Then $P(x) \in K[x]$ is $\mathbb{F}_q$-linear if

- P is additive
- $P(\zeta x) = \zeta P(x) \quad \forall \zeta \in \mathbb{F}_q$

$$P(\zeta x + \xi y) = \zeta P(x) + \xi P(y) \quad \forall \zeta, \xi \in \mathbb{F}_q$$

$x, y \in K$