Christelle Vincent!

Hello everyone!

Lecture 1: Non-Archimedean flds

2: Odds & Ends

3: Carlitz module

4: Drinfeld modules (Rank 2)

The Archimedean Property

Let K be a fld with an absolute value $|\cdot|$.

(no matter how small)

Then K is Archimedean if for any $x \neq 0, x \in K$

(no matter how large)

any $y \in K$ there is a positive integer n such that

$$|\underbrace{x+x+x+\dots+x}_{n \text{ times}}| > |y|$$

Example: $\mathbb{R}$, usual $|\cdot|$

OUR non-example

(an example of a non-Archimedean)

Let $\mathbb{F}_q$ be the finite fld with q elements

p prime $\quad q=p^r \quad r \geq 1$

$\mathbb{F}_q[T]$ the polynomial ring in T / $\mathbb{F}_q$

$\ni T^2+5T$

$$\mathbb{F}_q(T) = \left\{ \frac{a(T)}{b(T)} : a, b \in \mathbb{F}_q[T],\ b \neq 0 \right\}$$

the field!!

! If $r=1$ $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$

If $r\neq 1$ $\mathbb{F}_{p^r} \not\cong \mathbb{Z}/p^r\mathbb{Z}$

We need an absolute value on $\mathbb{F}_q(T)$

Def: Let K be a fld. An abs. value $|\cdot|$ on K is a function $|\cdot| : K \to \mathbb{R}_+$ s.t.

i) $|x|=0$ iff $x=0$

ii) $|x\cdot y| = |x|\cdot|y|$ $\forall x,y\in K$

iii) $|x+y| \leq |x|+|y|$ $\forall x,y\in K$ $\leftarrow$ Δ inequality

FOR $|\cdot|$ on $\mathbb{F}_q(T)$ maybe take

$|a| = \deg a$ if $a \in \mathbb{F}_q[T]$

$\left|\frac{a}{b}\right| = \deg a - \deg b$ if $\frac{a}{b} \in \mathbb{F}_q(T)$

<u>**But**</u> $a, b \in \mathbb{F}_q[T]$

$$\deg(a \cdot b) = \deg a + \deg b$$

This happens enough that we have a word for this!

Def: Let K be a fld. A valuation on K is a function

$$v: K \to \mathbb{R} \cup \{\infty\}$$

such that

i) $v(x) = \infty$ iff $x = 0$

ii) $v(x \cdot y) = v(x) + v(y) \quad \forall x, y \in K$

iii) $v(x+y) \geq \min(v(x), v(y)) \quad \forall x, y \in K$

equality if $v(x) \neq v(y)$

FOR US, we will let

$$v(a) = -\deg a \qquad a \in \mathbb{F}_q[T]$$

$$v\left(\frac{a}{b}\right) = \deg b - \deg a \qquad a, b \in \mathbb{F}_q[T]$$

Check that this <u>is</u> a valuation

i) set $v(0) = \infty$ $\quad v\left(\frac{a}{b}\right) = \infty \quad \Rightarrow \quad a = 0$ so $\frac{a}{b} = 0$

$\qquad = \deg b - \deg a$

ii) $v\left(\frac{a}{b} \cdot \frac{c}{d}\right) = \ldots\ldots = v\left(\frac{a}{b}\right) + v\left(\frac{c}{d}\right)$

↖ exercise

iii) only for $a, b \in \mathbb{F}_q[T]$ (leave case of $\frac{a}{b}, \frac{c}{d} \in \mathbb{F}_q(T)$)

$$v(a+b) = -\deg(a+b) \quad \text{(def)}$$

$$\deg(a+b) \leq \max(\deg a, \deg b)$$

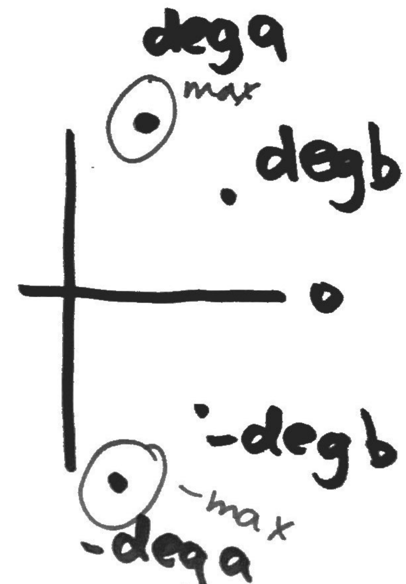
equality if $\deg a \neq \deg b$

$$-\deg(a+b) \geq -\max(\deg a, \deg b)$$

$$\| $$

$$\min(-\deg a, -\deg b)$$

$$v(a+b) \geq \min(v(a), v(b))$$



Def: On $\mathbb{F}_q(T)$, we will define

$$|x| = q^{-v(x)}$$

size of $\mathbb{F}_q$

$$\left|\frac{a}{b}\right| = q^{\deg a - \deg b}$$

Exercise: check that this is an absolute value

Proposition: This abs value is not Archimedean

$x = T \quad y = T^2 \quad |T + T + \dots + T| = q < |T^2| = q^2$

Proposition

An abs. value is non-Archimedean if and only if it satisfies, instead of the Δ inequality the stronger ultrametric inequality

$$|x+y| \leq \max(|x|, |y|)$$

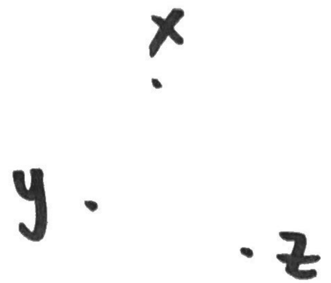
with equality if $|x| \neq |y|$

proof idea: ultra-metric $\Rightarrow$ non-Arch. is not bad

other implication is tricky

Proposition

In a non-Arch fld, every Δ is isosceles!



$$|x-y| = |x-z|$$

OR

$$|x-y| = |y-z|$$

OR

$$|y-z| = |x-z|$$

Def: $B(a,r) = \{ x \in K : |x-a| < r \}$

$r \in \mathbb{R}_+$

~~Def~~: In a non-Arch fld
Prop

i) $b \in B(a,r) \Rightarrow B(b,r) = B(a,r)$

Every pt in the circle is in the center!

ii) $B(a,r) \cap B(b,s) \neq \emptyset$

$\Rightarrow$ either $B(a,r) \subseteq B(b,s)$

or $B(b,s) \subseteq B(a,r)$

disjoint or completely inside each other

iii) $B(a,r)$ is both open & closed

Proposition

Let K be a complete non-Arch fld

$\sum_{n=0}^{\infty} x_n \qquad x_n \in K$ converges

iff $\quad \lim_{n\to\infty} |x_n| = 0$