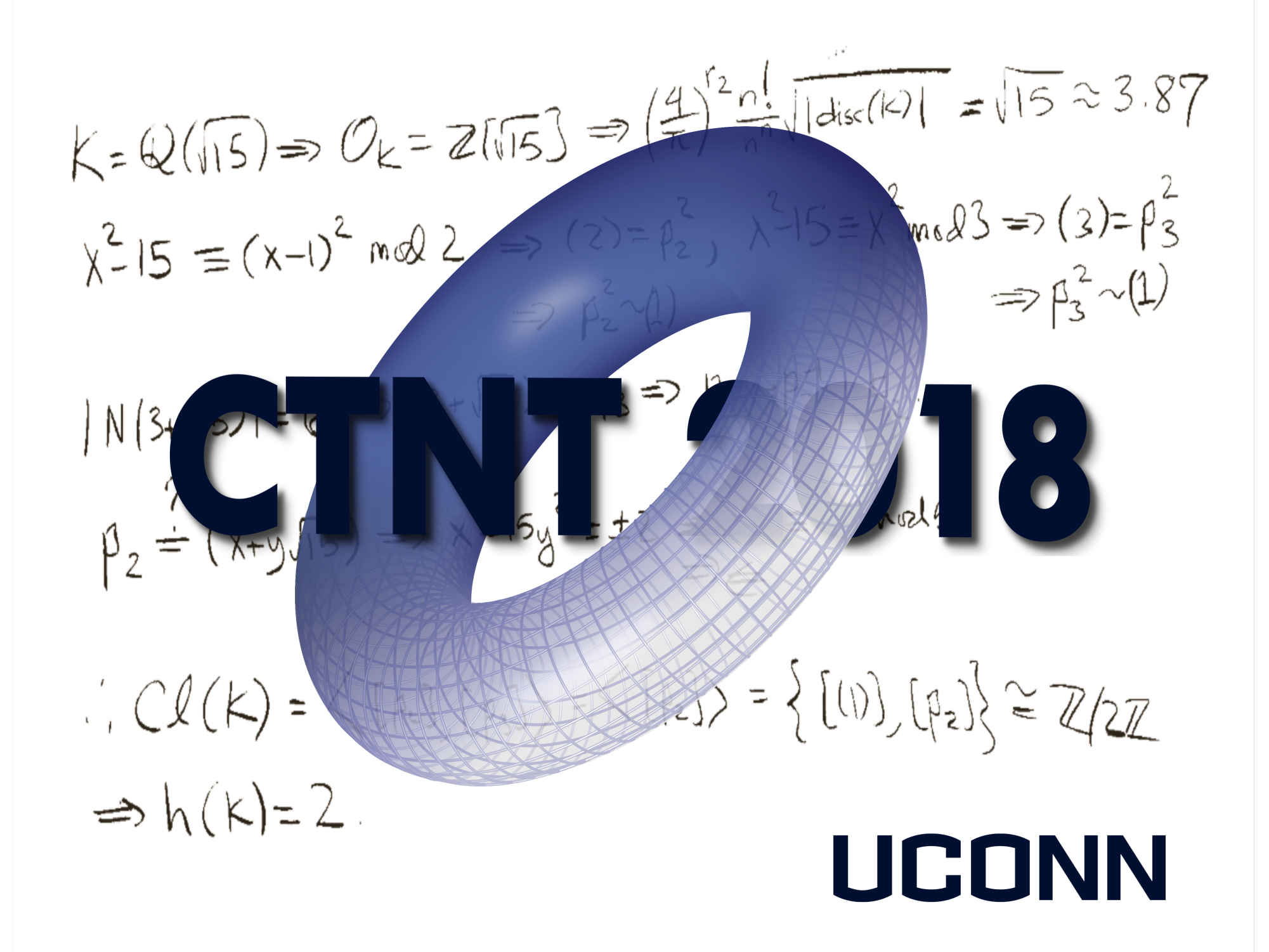
CTNT 2018
UCONN

Arithmetic Statistics
Lecture 3

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Department of Mathematics
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May 28th

CTNT 2018
Connecticut Summer School
in Number Theory

PREVIOUSLY...

We can define an **action of** $\mathrm{SL}(2,\mathbb{Z})$ **on Binary Quadratic Forms (BQFs)** by

$$M \cdot f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(M \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

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Proper equivalence leads to the *narrow* ideal class group of a ring of integers. **Wide equivalence** leads to the ideal class group.

PREVIOUSLY...

For example, $x^2 + xy + 5y^2$ is reduced, but $11x^2 + 5xy + y^2$ is NOT.

Gauss: Every equivalence class of BQFs (with disc $= d < 0$) contains a unique reduced representative $f(x, y) = ax^2 + bxy + cy^2$ with the property

$$-a < b \leq a \leq c, \text{ and } b \geq 0 \text{ if } a = c.$$

Also Gauss: there is an algorithm to find the reduced reprentative. Steps in the algorithm:

1. If $c < a$, then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$.
2. If $b > a$ or $b \leq -a$, then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$, such that $b' \equiv b \bmod 2a$ and $-a < b' \leq a$, with $b' = b - 2ka$.
3. If $c = a$ and $-a < b < 0$, then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$.

PREVIOUSLY...

Example

Let $d = -20$. The values of $g(x, y) = 2x^2 + 2xy + 3y^2$ are $(1/2)$·norms of elements in

$$(2, (2 + \sqrt{-20})/2) = (2, 1 + \sqrt{-5}) = \mathcal{P}_2,$$

which is a prime ideal above 2.

Is the ideal $\mathcal{P}_2 = (2, 1 + \sqrt{-5})$ principal? Is $\mathcal{P}_2 \sim \mathcal{O}_{K_d}$? If so:

- There is $\alpha \in K_d$ such that $\mathcal{P}_2 = \alpha \cdot \mathcal{O}_{K_d}$.
- $N(\tau) = N(\alpha)N(\gamma)$ for every $\tau \in \mathcal{P}_2$ and some $\gamma \in \mathcal{O}_{K_d}$.
- $2 = N(\mathcal{P}_2) = N(\alpha)N(\mathcal{O}_{K_d}) = N(\alpha)$, so $N(\alpha) = 2$.
- Thus, norms from $\mathcal{P}_2$ equal norms from $\alpha\mathcal{O}_{K_d}$ implies

$$2 \cdot g(x, y) = 2 \cdot f(x', y')$$

so $f(x, y) = x^2 + 5y^2$ and $g(x', y') = 2x'^2 + 2x'y' + 3y'^2$ represent the same numbers. Contradiction!!

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Upshot:

$$\begin{aligned}\left(3, \frac{2+\sqrt{-20}}{2}\right) &= \frac{3}{\frac{-2+\sqrt{-20}}{2}}\left(2, \frac{-2+\sqrt{-20}}{2}\right) \\ &= \frac{3}{\frac{-2+\sqrt{-20}}{2}}(2, -1+\sqrt{-5}) = \frac{3}{\frac{-2+\sqrt{-20}}{2}}(2, 1+\sqrt{-5})\end{aligned}$$

Note: What does the second step to reduce forms do to ideals?

- If $b > a$ or $b \leq -a$, then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$, such that $b' \equiv b \bmod 2a$ and $-a < b' \leq a$, with $b' = b - 2ka$.

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and these ideals are identical.

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The only values $d < 0$ with $h(d) = 1$, are

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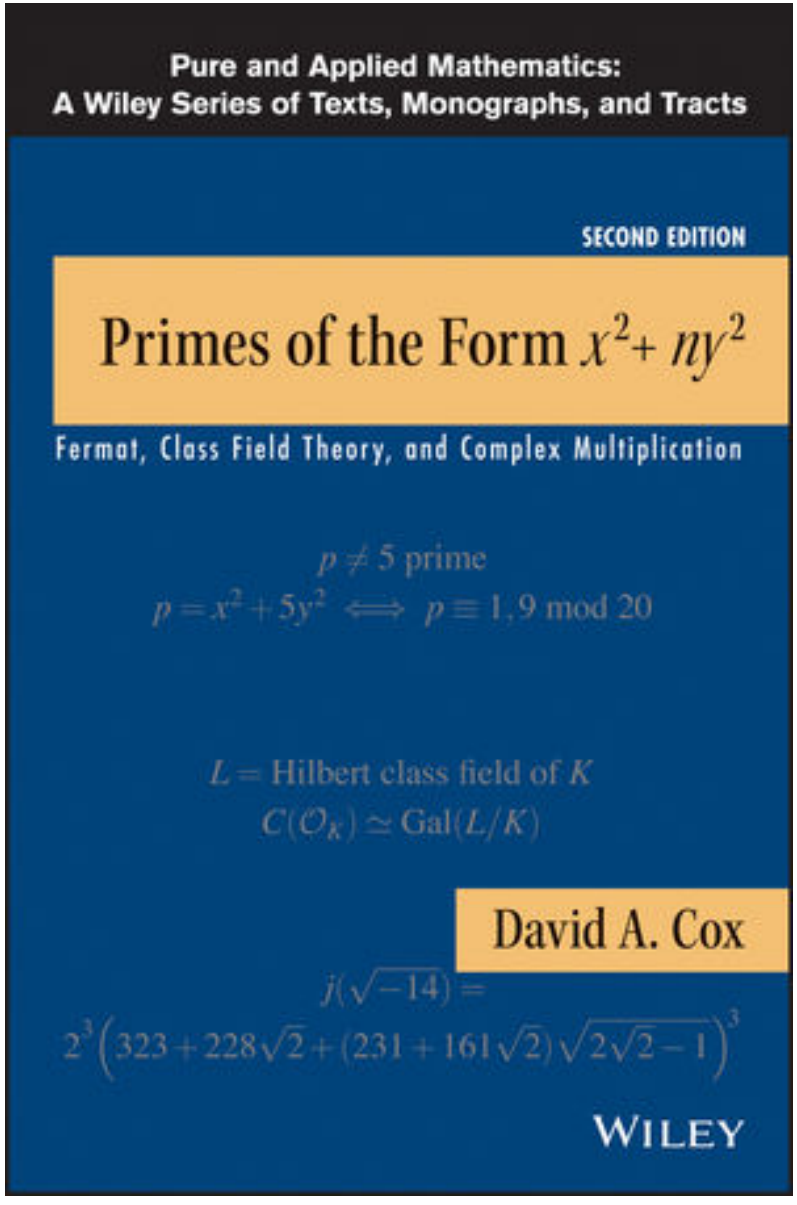
Gauss' conjecture: there are infinitely many number fields with $h = 1$.



A full treatment of binary quadratic forms and ideal class groups (§1 through §3).

Ideal Class Groups

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e.g., $\mathrm{Cl}(\mathbb{Q}(\sqrt{-5})) = \left\langle [\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}], [\mathcal{P}_2] \right\rangle \cong \mathbb{Z}/2\mathbb{Z}$.

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$$\mathcal{C} = \left\{ \mathrm{Cl}(\mathbb{Q}(\sqrt{d})) : d < 0 \text{ fundamental discriminant } (d \equiv 0, 1 \bmod 4) \right\}.$$

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Answer: NO. See genus theory.

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Example

Let $d = -20$ (so that $r = 2$). Then, we have $\mathrm{Cl}(\mathbb{Q}(\sqrt{-20})) \cong \mathbb{Z}/2\mathbb{Z}$.

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Conjecture (Cohen-Lenstra, 1984)

The sequence of groups $H_d^{\neq 2}$ ordered by $|d|$ behaves like a "random sequence of finite abelian groups of odd order".

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What group is more likely: $\mathbb{Z}/p^3\mathbb{Z}$, $(\mathbb{Z}/p\mathbb{Z})^3$, or $\mathbb{Z}/p\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^2$?

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Cohen-Lenstra: Let p be a prime. Assume that we have a "natural" unbiased stochastic process producing finite abelian p-groups. If we fix a finite abelian group G, then the probability that an output of the process is isomorphic to G is inversely proportional to the size of the automorphism group $\mathrm{Aut}(G)$.

Theorem

Let $k \geq 0$, $e_1 > \ldots > e_k > 0$, and $r_i > 0$ for $i = 1, \ldots, k$. Let

$$G = \prod_{i=1}^{k} (\mathbb{Z}/p^{e_i}\mathbb{Z})^{r_i}.$$

Then,

$$\#\,\mathrm{Aut}(G) = \left(\prod_{i=1}^{k} \left(\prod_{s=1}^{r_i} (1 - p^{-s}) \right) \right) \cdot \left(\prod_{1 \leq i,j \leq k} p^{\min(e_i, e_j) r_i r_j} \right).$$

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Let $G = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$, so $k = 2$, $e_1 = 1$, $e_2 = 2$, $r_1 = 1$, $r_2 = 1$. Thus,

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Example

Let $p = 5$. Then:

$$\begin{aligned} \#\operatorname{Aut}(\mathbb{Z}/p^3\mathbb{Z}) &= (p-1)p^2 = 100 \\ \#\operatorname{Aut}((\mathbb{Z}/p\mathbb{Z})^3) &= (p-1)(p^2-1)(p^3-1)p^3 = 1488000 \\ \#\operatorname{Aut}((\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})) &= (p-1)^2p^3 = 2000. \end{aligned}$$

Cohen-Lenstra: Let p be a prime. Assume that we have a "natural" unbiased stochastic process producing finite abelian p-groups. If we fix a finite abelian group G, then the probability that an output of the process is isomorphic to G is inversely proportional to the size of the automorphism group $\mathrm{Aut}(G)$.

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Formally: Let $\mathcal{G}_p$ be the set of all finite abelian p-groups. The Cohen-Lenstra weight ω is the measure on the set $\mathcal{G}_p$ such that

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Therefore:

$$\begin{aligned}
P(\mathbb{Z}/p^3\mathbb{Z}) &= \frac{1/100}{1.315213\ldots} = 0.0076\ldots \text{ or } 0.76\% \\
P((\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})) &= \frac{1/2000}{1.315213\ldots} = 0.00038\ldots \text{ or } 0.038\%. \\
P((\mathbb{Z}/p\mathbb{Z})^3) &= \frac{1/1488000}{1.315213\ldots} = 0.0000005109\ldots \text{ or } 0.00005109\%.
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WHY THIS MEASURE???

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Why would the Cohen-Lenstra measure be the correct measure?

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This is **unnatural** because, for one thing, we would be assuming the structure theorem of finite abelian groups.

Instead, let us model random generators and relations.

Start with $r \geq 1$ generators $\mathbb{Z}^r = \langle g_1, \ldots, g_r \rangle$.

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In matrix form: $R = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,r} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ e_{r,1} & e_{r,2} & \cdots & e_{r,r} \end{pmatrix}$

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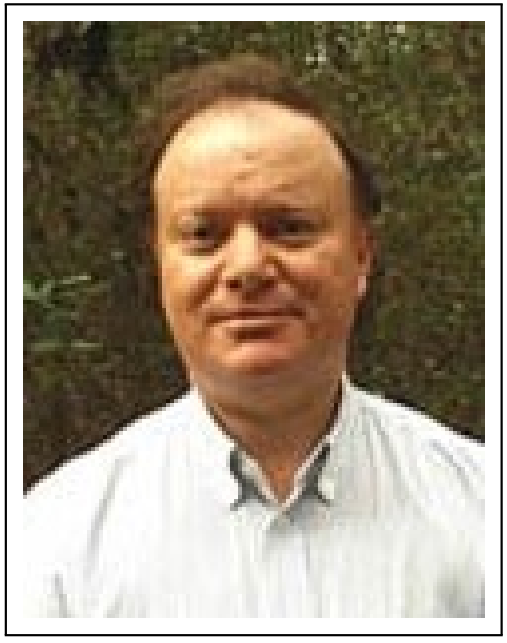
Moreover, $\mathbb{Z}_p$ is compact, it has a Haar measure (normalized to have total volume 1), and so $\mathbb{Z}_p^{r \times r}$ inherits a Haar measure from $\mathbb{Z}_p$.



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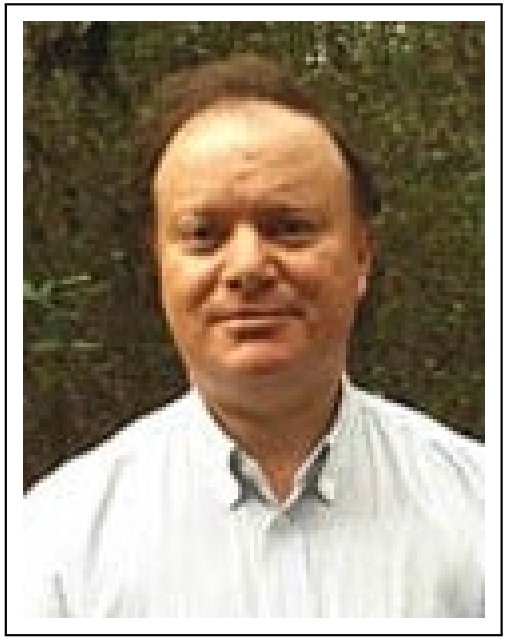
Lawrence Washington

Theorem (Friedman, Washington, 1987)

For a randomly chosen matrix $R \in \mathbb{Z}_p^{r \times r}$ w.r.t. the Haar measure,



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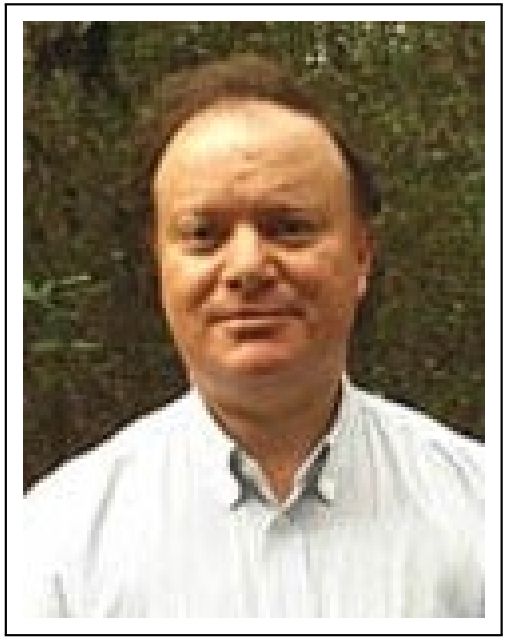

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1. $P(R \text{ has full rank }) = 1$ *for all* $r > 0$.
2. *For any finite abelian p-group G we have*

$$P(\mathrm{Coker}(R) \cong G) \longrightarrow P(G)$$

as $r \to \infty$, *where P is the* **Cohen-Lenstra probability measure**.

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For quadratic fields, there is extensive data (with hundreds of millions of class groups) to support the Cohen-Lenstra heuristics. See Stephens and Williams [Stephens and Williams 88]; Jacobson, Lukes, and Williams [Jacobson et al. 95]; and Jacobson [Jacobson 98].

How about **real quadratic fields**? The Cohen-Lenstra heuristics for a real quadratic field $\mathbb{Q}(\sqrt{d})$ suggests the following:

Let $k \geq 1$ be an odd number. Then, the probability that the odd part of $h(\mathbb{Q}(\sqrt{d}))$, denoted by $h^*(\mathbb{Q}(\sqrt{d}))$, equals k is given by

$$P(h^*(\mathbb{Q}(\sqrt{d})) = k) = \frac{C \cdot \lambda(k)}{k}$$

where $C = 0.754458173\ldots$, and

$$\lambda(k)^{-1} = \prod_{p^\alpha || k} p^\alpha (1 - p^{-1})(1 - p^{-2}) \cdots (1 - p^{-\alpha}).$$

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If we assume that the subfamily $\mathbb{Q}(\sqrt{p})$, over primes p, behaves similarly, then $h^* = h$, and

$$P(h(\mathbb{Q}(\sqrt{p})) = 1) = C = 0.754458173\ldots.$$

THANK YOU

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"*If by chance I have omitted anything*
more or less proper or necessary,
I beg forgiveness,
since there is no one who is without fault
and circumspect in all matters."

Leonardo Pisano (Fibonacci), *Liber Abaci*.