Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the mind will never penetrate. L. Euler

Resources and references can be found here https://alozano.clas.uconn.edu/arithmetic-statistics/

Exercise 1. Show that $\log(n!) \sim n \cdot \log n$, or, in other words, $\lim_{n\to\infty} \frac{\log(n!)}{n \cdot \log n} = 1$. (Hint: interpret $\log(n!) = \log(1) + \log(2) + \cdots + \log(n)$ as a Riemann sum.)

Exercise 2. Let $a_1 = 0, a_2, \ldots, a_k$ be integers such that there is no prime p with the property that the set $\{a_i \bmod p\}$ covers all the values modulo p.

- Use a computer (and Sagemath, Magma, or any other software) to find an admissible 447-tuple with $a_1 = 0, \ldots, a_{447} \leq 3159$.
- Is 447 the smallest value of k such that there exists a k-tuple $a_1 = 0, \ldots, a_k \leq y$ with $\pi(y) < k$? Here $\pi(X)$ is the prime number counting function. (Any such k-tuple would show that the Hardy-Littlewood conjecture on prime constellations implies that their 2nd conjecture is false.)

Exercise 3. An odd prime p is called a Wieferich prime (in base 2) if $2^{p-1} \equiv 1 \bmod p^2$. It has been conjectured that the number of Wieferich primes $p \leq X$ is approximately $\log(\log(X))$. Give a plausible heuristic argument in favor of this conjecture.

Exercise 4. A prime p is called a Sophie Germain prime if $q = 2p + 1$ is also prime. Give a reasonable asymptotic for the number of Sophie Germain primes $p \leq X$, and a heuristic argument to support your conjecture. Can you provide data that supports your claims?

Exercise 5. Every odd prime number is $\equiv 1, 3, 5$, or $7 \bmod 8$. Is any of these equivalence classes more or less common than the others among the primes up to a given bound X? Provide a table of data that supports your observations.

Exercise 6. Every odd prime number is $\equiv 1, 2, 4, 5, 7$, or $8 \bmod 9$. Is any of these equivalence classes more or less common than the others among the primes up to a given bound X? Provide a table of data that supports your observations.

Exercise 7. Use Gauss' algorithm to find a reduced form equivalent to $3x^2 + 9xy + 8y^2$.

Exercise 8. Use Gauss' algorithm to find a reduced form equivalent to $6x^2 - 9xy + 4y^2$.

Exercise 9. Are $3x^2 + 9xy + 8y^2$ and $6x^2 - 9xy + 4y^2$ binary quadratic forms that are $\mathrm{SL}(2, \mathbb{Z})$-equivalent?

Exercise 10. The number of classes in $\mathrm{BQFs}(d)/\mathrm{SL}(2, \mathbb{Z})$ is denoted by $h(d)$ and called the class number of binary quadratic forms of discriminant d. Show that $h(-3) = h(-4) = 1$.

Exercise 11. Compute $h(-15)$.

Exercise 12. The first negative fundamental discriminant $d < 0$ with $h(d) = 3$ is $d = -23$. Find three inequivalent reduced quadratic forms of discriminant -23.

Exercise 13. Let $\gcd(a, b, c) = 1$, let p be a prime, and put $d = b^2 - 4ac$. Show the following:

1. If $p = am^2 + bmn + cn^2$ for integers m, n, then $d \equiv \square \bmod 4p$.
2. If d is a square mod $4p$, then there exists a binary quadratic form of discriminant d that represents p.

Exercise 14. Let $K = \mathbb{Q}(\sqrt{-20})$, let $\mathcal{O}_K$ be its ring of integers, and let $I = (23, 8 + \sqrt{-5})$ and $J = (29, 13 + \sqrt{-5})$. Decide whether I and J are principal ideals.

Exercise 15. Compute formulas for the size of the automorphism group of all finite abelian groups of order p^2, where p is prime. Evaluate your formulas at $p = 3$.

Exercise 16. Let p be a prime and let $\omega(\mathcal{G}_p) = \prod_{i=1}^{\infty}(1 - p^{-i})^{-1}$. Approximate the values $\omega(\mathcal{G}_p)$ for $p = 2, 3, 5$, and 7.

Exercise 17. Use a database of number fields (and Sagemath or Magma) to extract a database of imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, and class groups $H_d = \mathrm{Cl}(\mathbb{Q}(\sqrt{-d})))$, and 3-parts of H_d, i.e., a database of $H_d[3^\infty]$.

1. Find the proportion of values of $-d$ such that $H_d \cong G$ for each of $G = \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z}$, and $(\mathbb{Z}/3\mathbb{Z})^2$.
2. Compare the values you found in the database with the conjectural values that come from the Cohen-Lenstra heusristic.

Exercise 18. Let $p > 2$ be a prime. For each part below, find a matrix $R \in \mathbb{Z}_p^{3\times 3}$ such that

1. $\mathbb{Z}_p^3/\operatorname{Col}(R) \cong \mathbb{Z}/p\mathbb{Z}$.
2. $\mathbb{Z}_p^3/\operatorname{Col}(R) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.
3. $\mathbb{Z}_p^3/\operatorname{Col}(R) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.
4. $\mathbb{Z}_p^3/\operatorname{Col}(R) \cong \mathbb{Z}/p^3\mathbb{Z}$.
5. $\mathbb{Z}_p^3/\operatorname{Col}(R) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$,

where $\operatorname{Col}(R)$ is the $\mathbb{Z}_p$-module generated by the columns of R.

Exercise 19. Use a database of elliptic curves to compute the proportion of elliptic curves $E/\mathbb{Q}$ with $E(\mathbb{Q})_{\mathrm{tors}} \cong \mathbb{Z}/5\mathbb{Z}$ of naive height up to a bound X (of your choice), and compare the value you obtain to the Harron-Snowden result on the density of elliptic curves with prescribed torsion.