Origami Galois representations

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Connecticut Conference on Elliptic Curves and Modular Forms, and Related Topics
University of Connecticut

August 18, 2016

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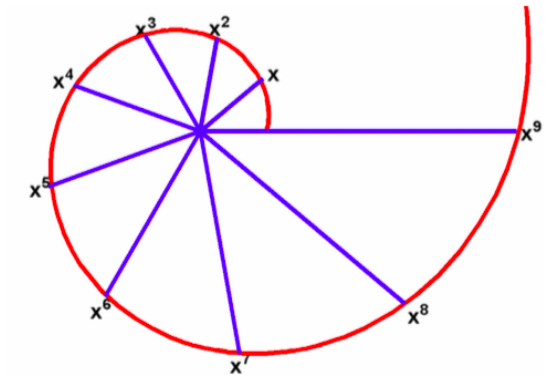
Let $\phi : X \to Y \quad X = Y = \mathbb{G}_m =$ multiplicative group $= \mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$.
$\phi : Y \to X$
$\phi : x \mapsto x^N$



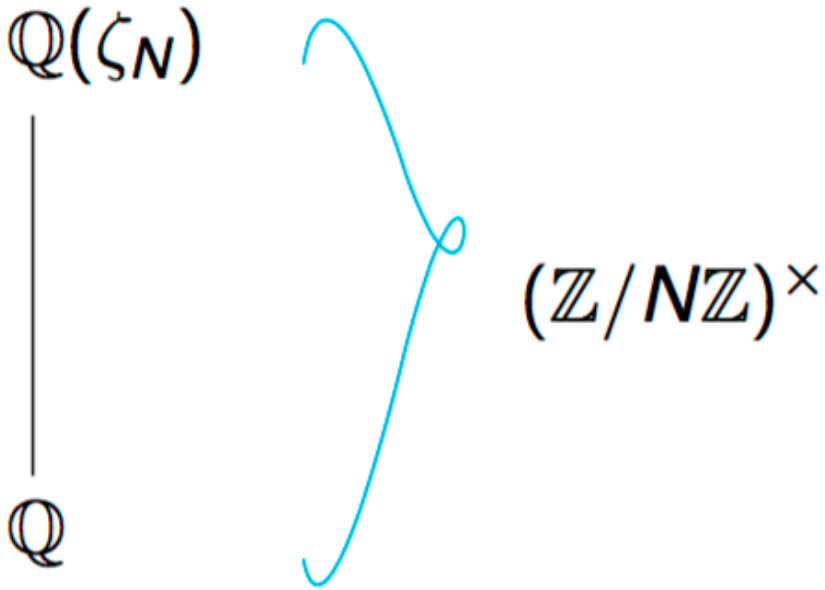
Fix a rational point $p \in \mathbb{G}_m$ (thought of in X, the target of the map ϕ).

Consider the set $V = \phi^{-1}(p) = \{x \in \mathbb{G}_m | \quad \phi(x) = p\}$, i.e. $\{x \in \mathbb{G}_m | \quad x^N = p\}$.

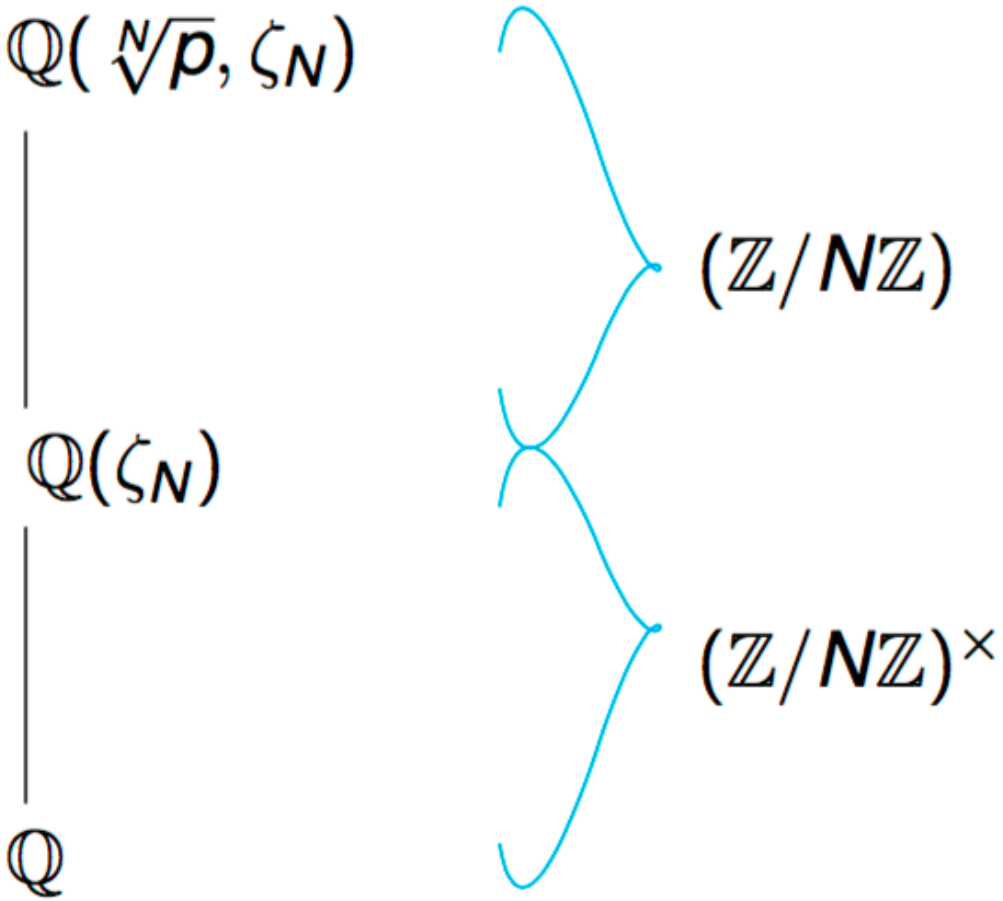
$$f_p(x) = x^N - p$$

First, consider the case that $p = 1$. Then V is the set of solutions to $f_p(x) = x^N - 1$. These are the N^{th} roots of unity.

$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times$. Let i, N be relatively prime. Then $\sigma_i : \zeta_N \mapsto \zeta_N^i$.



When $f_p(x) = x^N - p$ is irreducible, the picture becomes the following:



- $\mathrm{Gal}(sf(x^N - p)/\mathbb{Q})$ is a subgroup of $\mathrm{AGL}_1(\mathbb{Z}/N\mathbb{Z})$.
- There is a Galois representation

$$\rho_{N,p} : G_{\mathbb{Q}} \to \mathrm{AGL}_1(\mathbb{Z}/N\mathbb{Z})$$

- This is given by $\sigma \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ such that $\sigma(\zeta_N) = \zeta_N^a$ and $\frac{\sigma(\sqrt[N]{p})}{\sqrt[N]{p}} = \zeta_N^b$ with $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $b \in \mathbb{Z}/N\mathbb{Z}$.

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Let E be an elliptic curve over $\mathbb{Q}$. Fix a positive integer N. We define multiplication by N on E, denoted $[N]$ to be adding a point to itself N times. We define the N-division points of E:

$$E[N] = \left\{ P \in E(\overline{\mathbb{Q}}) : [N]P = \mathcal{O} \right\}.$$

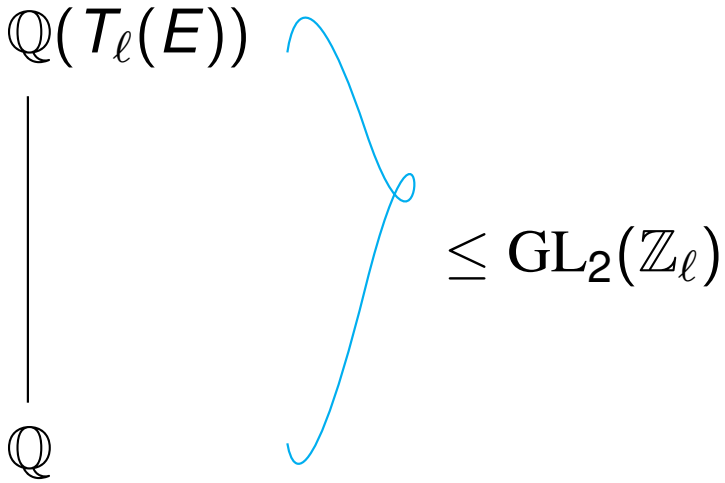
Facts:

- The N-division points form a group that is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$. For example, $E[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$, a Klein 4-group.
- The Galois group $G_{\mathbb{Q}}$ permutes the N-division points. We denote the representation by $\overline{\rho}_{E,N}$.

We will write $\mathbb{Q}(E[N])$ to mean the field obtained by adjoining all of the coordinates of the N-division points of E to $\mathbb{Q}$.

Fix a prime ℓ and define the Tate module to be he inverse limit $T_\ell(E) = \varprojlim_n E[\ell^n]$

There is a representation $\rho_{E,\ell}$ from $G_{\mathbb{Q}}$ acting on $T_\ell(E)$ over $\mathbb{Q}$ into $\mathrm{Aut}(\mathbb{Z}_\ell^2)$, which after a choice of basis can be identified with $\mathrm{GL}_2(\mathbb{Z}_\ell)$.



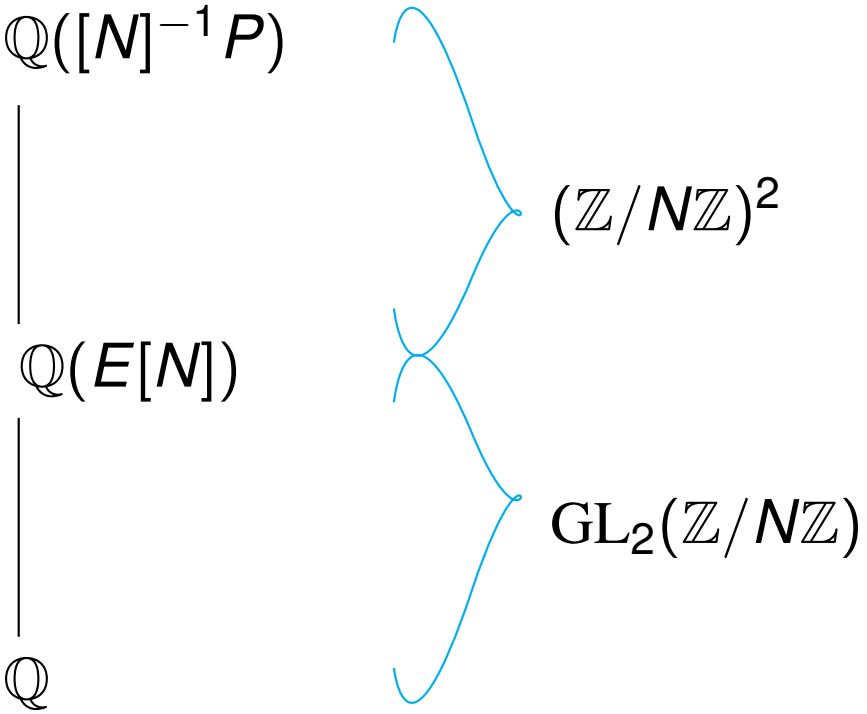
- Consider the case that E is an elliptic curve over $\mathbb{Q}$ without complex multiplication. In this case, a result of Serre shows that the image of $\rho_{E,\ell}$ has finite index in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for all ℓ. Also, the representation is surjective except for a finite set of primes S_E.
- $\ell = 2$ The map $\rho_{E,2}$ is surjective if and only if $\overline{\rho}_{E,8}$ is surjective.
- $\ell = 3$ The map $\rho_{E,3}$ is surjective if and only if $\overline{\rho}_{E,9}$ is surjective.
- $\ell \geq 5$ The map $\rho_{E,\ell}$ is surjective if and only if $\overline{\rho}_{E,\ell}$ is surjective.

Let E be given by $y^2 = x^3 + Ax + B$. Fix a point $P \in E(\mathbb{Q})$ given by $P = (z : w : 1)$. Consider the set

$$V = [N]^{-1}P = \left\{ Q \in E(\overline{\mathbb{Q}}) | \quad [N]Q = P \right\}.$$

For example, when $P = \mathcal{O}$, this set is the set of N-division points.

This is no longer a group in general, but we can still adjoin the coordinates of such points to $\mathbb{Q}$ and find the Galois group of the extension.


$\mathbb{Q}([N]^{-1}P)$
$\mathbb{Q}(E[N])$
$\mathbb{Q}$
$(\mathbb{Z}/N\mathbb{Z})^2$
$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$

The Galois group of $\mathbb{Q}([N]^{-1}P)$ over $\mathbb{Q}$ is a subgroup of the affine general linear group

$$1 \to (\mathbb{Z}/N\mathbb{Z})^2 \to \mathrm{AGL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

e.g. for $N = 2$

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to S_4 \to S_3 \to 1$$

$$\left\{ \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, d, e, f \in (\mathbb{Z}/N\mathbb{Z}) \text{ and } ad - bc \neq 0 \right\}$$

Again, fixing ℓ and taking an inverse limit yields a representation

$$\rho_{E,\ell,P} : G_{\mathbb{Q}} \to \mathrm{Aut}(\mathbb{Z}_\ell^2) \simeq \mathrm{AGL}_2(\mathbb{Z}_\ell)$$

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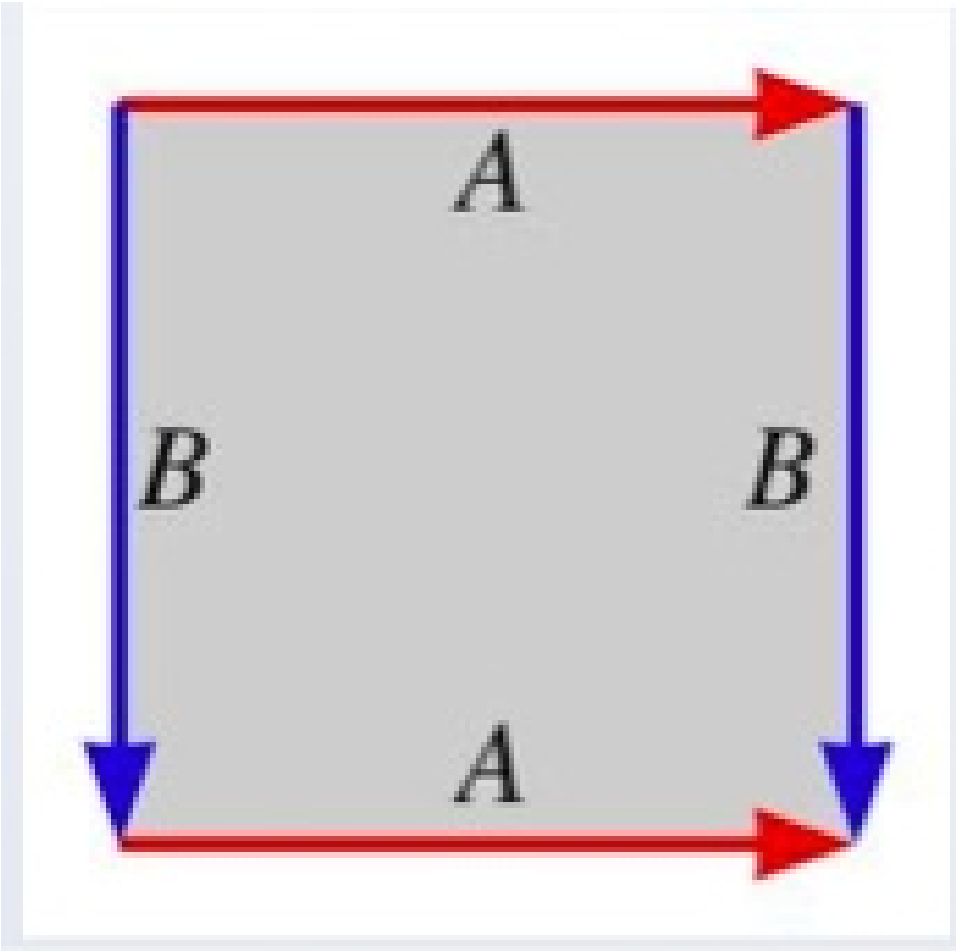
There is an exact sequence (due to Grothendieck and others) of fundamental groups

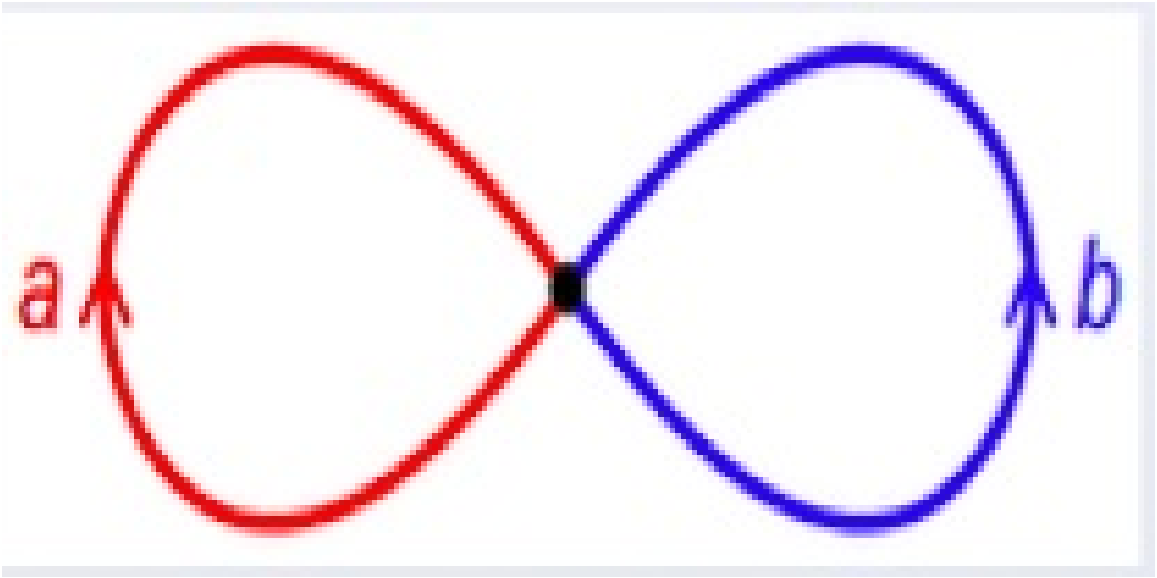
$$1 \to \pi_1(X_{\overline{\mathbb{Q}}}) \to \pi_1(X) \to G_{\mathbb{Q}} \to 1$$

which group-theoretically gives rise to a representation

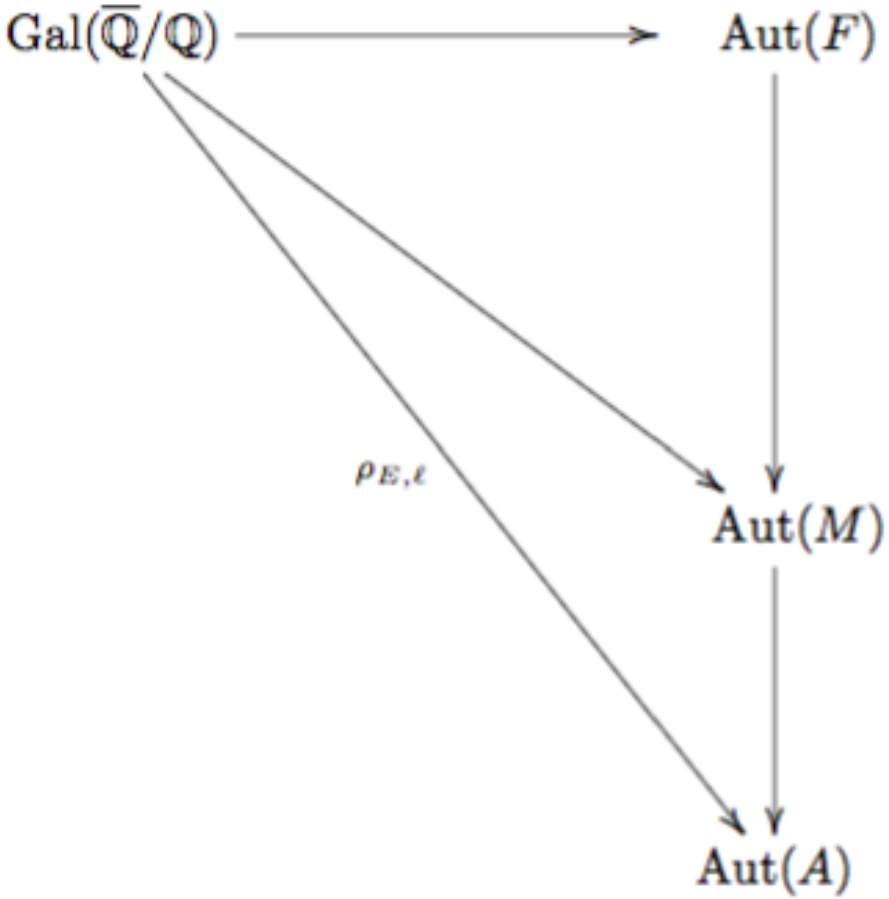
$$\rho_X : G_{\mathbb{Q}} \to \mathrm{Aut}(\pi_1(X_{\overline{\mathbb{Q}}})).$$

There are comparison theorems that allow one to determine the geometric fundamental group by considering the variety X over $\mathbb{C}$.


A
B
B
A


a
b

Any two-generated group should arise as the group of deck transformations of some cover of a once-punctured elliptic curve. So far, we have only encountered the abelian deck groups $\mathbb{Z}_\ell^2$. We will restrict to studying pro-ℓ groups (for now pro-2) and look for some barely non-abelian examples.


$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
$\mathrm{Aut}(F)$
$\rho_{E,\ell}$
$\mathrm{Aut}(M)$
$\mathrm{Aut}(A)$

Define

$$M_2 = \left\{ a, b, c | a^4, b^4, c^2, c = aba^{-1}b^{-1}, ac = ca, bc = cb \right\}$$

Then M_2 is a group of order 32 whose abelianization is $(\mathbb{Z}/4\mathbb{Z})^2$. Also, Q_8, the finite group of order 8 of the quaternions is a characteristic quotient of M_2.

Theorem (D., Goins)

Suppose that

$$\overline{\rho}_{E,8} \text{ and } \overline{\rho}_{E,M_2}$$

are both surjective. Then, the representation to $\mathrm{Aut}(M)$ *is surjective.*

Let E be an elliptic curve over $\mathbb{Q}$.

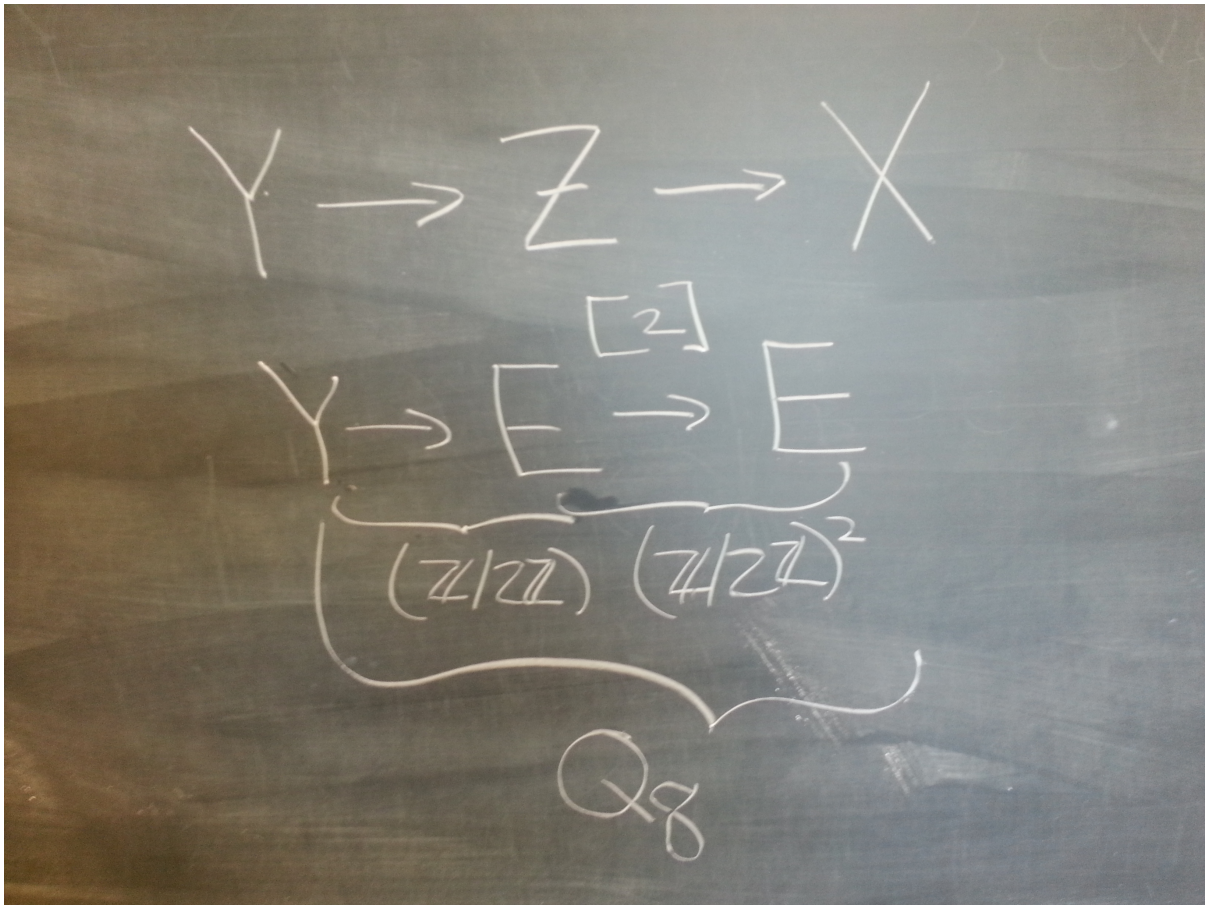
Definition

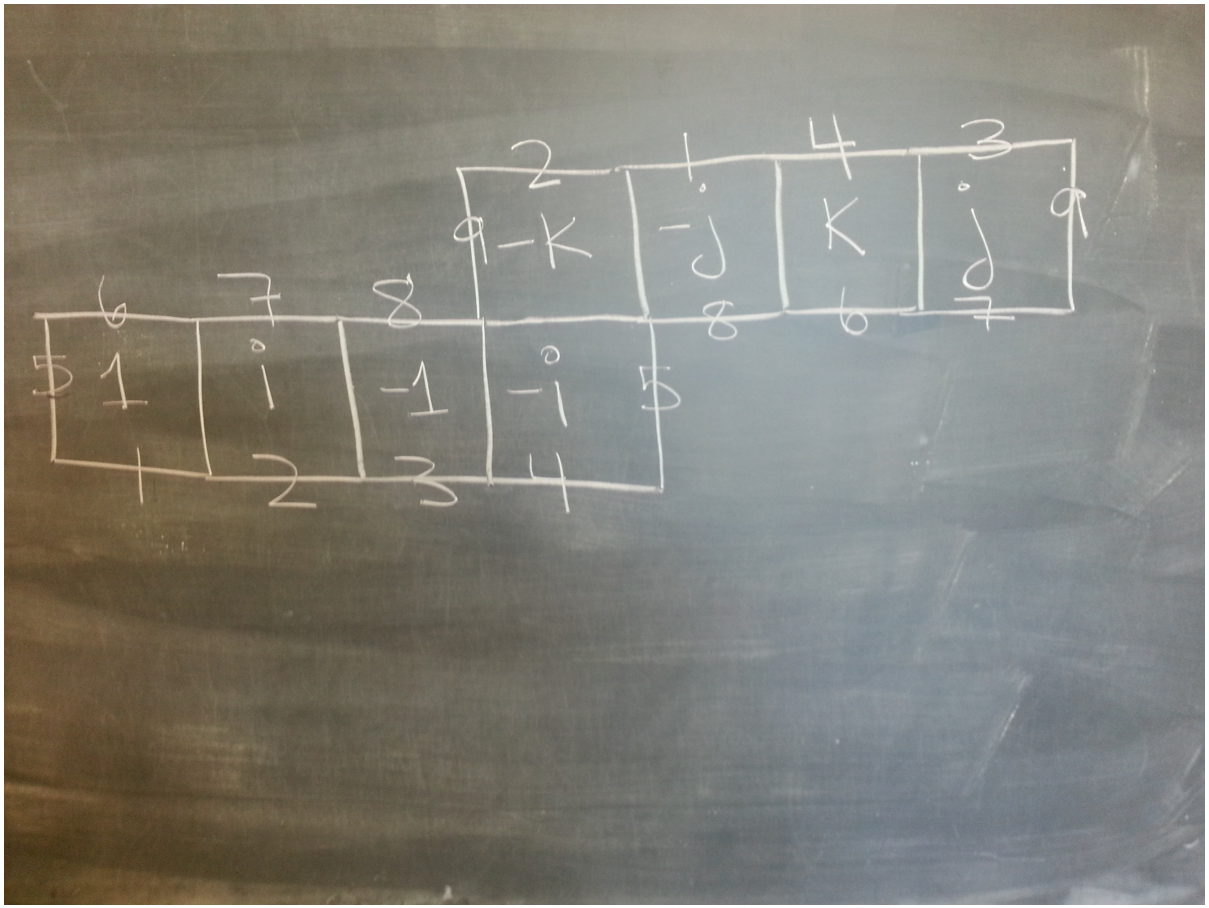
An origami is a pair (C, f) where C is a curve and $f : C \to E$ is a map branched above at most one point.

Definition

A deck transformation or automorphism of a cover $f : C \to E$ is a homeomorphism $g : C \to C$ such that $f \circ g = f$.

Each deck transformation permutes the elements of each fiber. This defines a group action of the the deck transformations on the fibers.


Y → Z → X
[2]
Y → E → E
(Z/2Z) (Z/2Z)²
Q8


2
1
4
3
9
-k
-j
k
j
9
6
7
8
8
6
7
5
1
i
-1
-i
5
4
2
3
4

“An extraordinary origami curve" by Herrlich and Schmithüesen

$$f = 8$$

$$e = \frac{8 \cdot 4}{2} = 16$$

$$v = \frac{8 \cdot 4}{4 \cdot 2} = 4$$

Formula for Euler characteristic:

$$2 - 2g = v - e + f$$

$$\implies g = 3$$

Riemann-Hurwitz formula

$$f : Y \to X$$

Then,

$$2g(Y) - 2 = \deg(f) \cdot (2g(Z) - 2) + \sum_{z \in Z} (e_z - 1)$$

Applying Riemann-Hurwitz with $g(Y) = 3$, $g(X) = 1$, we see that there are 4 points ramified in Y, each with ramification degree 2.

In fact, Herrlich and Schmithüesen give that the map $Y \to Z$ is given by $(x, y) \mapsto (x, y^2)$ and $Y : y^4 = x^3 + Ax + B$. This is an example of a superelliptic curve.

Let $\Delta = -16(4A^3 + 27B^2)$ and w_2, ϕ_2, ψ_2 be the usual division polynomials of elliptic curves as defined e.g. by Silverman. Taking the resultant of $\phi_2 - z\psi_2^2$ and $w_2 - w\psi_2^3$ yields $y^4 - 8wy^3 + 6(2Az + 3B)y^2 - \Delta = 0$. Plugging in y^2 for y yields

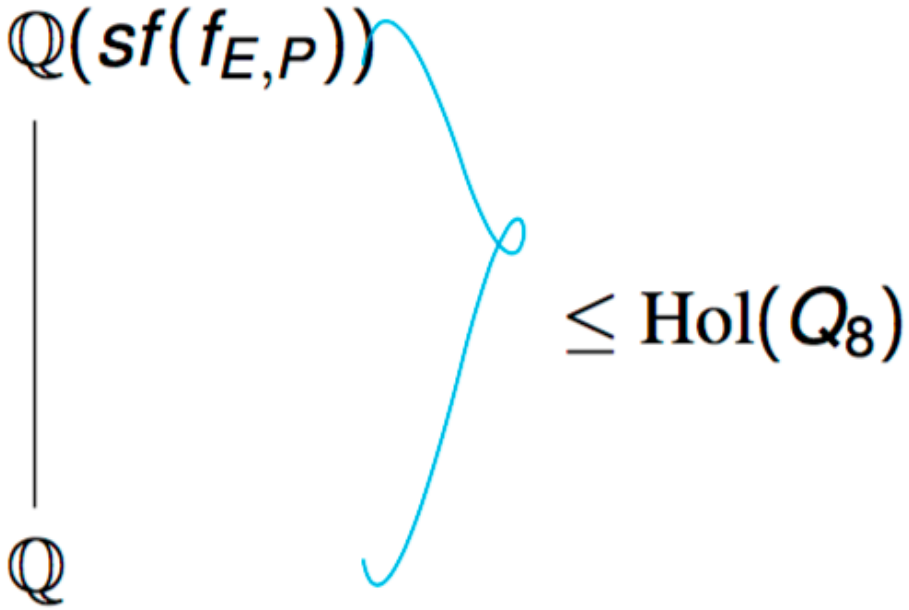
$$f_{E,Q_8,P} = y^8 - 8wy^6 + 6(2Az + 3B)y^4 - \Delta$$

Theorem (D., Goins)

Fix a rational point $P = (z : w : 1) \in E(\mathbb{Q})$. Consider the extension $F_P = \mathbb{Q}(sf(f_{E,Q_8,P}))/\mathbb{Q}$ given by the splitting field of the polynomial $f_{E,Q,8,P}$. Then

$$\mathrm{Gal}(F_P/\mathbb{Q}) \leq \mathrm{Hol}(Q_8)$$

with equality if the polynomial is irreducible.


$\mathbb{Q}(sf(f_{E,P}))$
$\mathbb{Q}$
$\leq \mathrm{Hol}(Q_8)$

$$1 \to Q_8 \to \mathrm{Hol}(Q_8) \to \mathrm{Aut}(Q_8) \simeq S_4 \to 1$$

The group $\mathrm{Hol}(Q_8)$ is a specific group of order 192.

There is a quotient from $\rho_{E,Q_8,P} : G_{\mathbb{Q}} \to \mathrm{Hol}(Q_8)$ to $\rho_{E,Q_8} : G_{\mathbb{Q}} \to \mathrm{Aut}(Q_8)$

Theorem (D., Goins)

The image of the quotient is given by the Galois group of the splitting field of $x^4 - 4\Delta x - 12A\Delta$.

Remark: This quartic polynomial defines the unique S_4-representation contained inside of the 4-division representation of E. (See, for instance, Adelmann's "The decomposition of primes in torsion point fields".)

Thank you! Questions?