Geometric and p-adic aspects of mock modular forms

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What is a mock modular form?

Definition

A harmonic Maass form of level $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and weight $k \in \mathbb{Z}$ is a smooth function $F : \mathfrak{h} \to \mathbb{C}$ such that

(i) $F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k F(\tau), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

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A mock modular form is the holomorphic part F^+ of a harmonic Maass form.

Some geometric notation

Set $\Gamma = \Gamma_0(N)$ (or $\Gamma_1(N)$), $K \subseteq \mathbb{C}$.

- $Y = Y_\Gamma =$ modular curve over K, $X = X_\Gamma = Y \cup C$.

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The weight k parabolic cohomology ($k \geq 2$) is the K-vector space

$$\mathbb{H}^1_{\mathrm{par}}(X, \nabla_{k-2}) := \mathbb{H}^1(\mathcal{H}_{k-2} \otimes \mathcal{I}_C \xrightarrow{\nabla_{k-2}} \mathcal{H}_{k-2} \otimes \Omega^1_X),$$

Parabolic cohomology

- $S_k(\Gamma, K)$ = cusp forms with coefficients in K.
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Theorem (Shimura, Deligne)

$$\mathbb{H}^1_{\mathrm{par}}(X_{\mathbb{C}}, \nabla_{k-2}) \simeq S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})}$$

Geometric point of view

- $f \in S_k(\Gamma, K)$ a newform, $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$ the isotypical component.

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Unknown even for $k = 2$ (Katz).

Rigid analytic spaces

$p > 3$ a prime, $p \nmid N$, $K_p = p$-adic completion of K.

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- Let $W_1 = X_{(p^{-p/p+1})}$ and $W_2 = X_{(p^{-1/p+1})}$. There are p-adic operators

$$U = U_p : H^0(W_2, \underline{\omega}^k) \longrightarrow H^0(W_1, \underline{\omega}^k) \subseteq H^0(W_2, \underline{\omega}^k)$$

$$V = V_p : H^0(W_1, \underline{\omega}^k) \longrightarrow H^0(W_2, \underline{\omega}^k)$$

p-stabilization

Theorem

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), K)$ *be a newform and let*

$$T^2 - a_p T + \chi(p)p^{k-1} = (T - \beta)(T - \beta')$$

be the p*-th Hecke polynomial of* f*. Then the overconvergent modular forms*

$$f_\beta := f - \beta' V(f), \quad f_{\beta'} := f - \beta V(f)$$

in $H^0(W_2, \underline{\omega}^k)$ *are* U*-eigenvectors with eigenvalues* β *and* β'*, respectively.*

Cohomology

Theorem (C.-Castella)

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), K)$ be a newform of weight $k \geqslant 2$, and let β and β' be the roots of $T^2 - a_p T + \chi(p) p^{k-1}$, ordered so that $v_p(\beta) \leqslant v_p(\beta')$. Assume that the following two conditions hold:

(i) $\beta \neq \beta'$.

(ii) $v_p(\beta') \neq k-1$.

Then $\{[f], [V(f)]\}$ is a basis for $\mathbb{H}^1_{\mathrm{par}}(X_{K_p}, \nabla_{k-2})_f$.

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- Idea: $[f_\beta], [f_{\beta'}]$ form a basis.
- Compare to: $\mathbb{H}^1_{\mathrm{par}}(X_{\mathbb{C}}, \nabla_{k-2})_f = \mathbb{C}\, f \oplus \mathbb{C}\, \bar{f}$

p-adic HMF and MMF

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- $F_p^+ =$ Eichler integral of $\phi - t_1 f$ (a p-adic mock modular form).

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- Emerton: Variational Hodge Conjecture $\Rightarrow$ Coleman Conjecture.

p-adic properties of MMF

Theorem (Guerzhoy-Kent-Ono)

Let $\alpha \in \mathbb{C}$ be such that $\alpha - c^+(1) \in K$. Then the coefficients of

$$\mathcal{F}_\alpha := F^+ - \alpha E_f := \sum_{n \gg -\infty} c^+(n) q^n - \alpha \sum_{n=1}^{\infty} a(n) n^{1-k} q^n$$

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Proof.

$F^+ = E_\phi - s_1 E_f$, $s_1 \in \mathbb{C}$. □

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Assume that $v_p(\beta) < v_p(\beta')$ and that $v_p(\beta') \neq k-1$. Then

$$\lim_{w \to +\infty} \frac{U^w D^{k-1}(\mathcal{F}_{c^+(1)})}{c_{c^+(1)}(p^w)} = f_\beta,$$

where we write $D^{k-1}(\mathcal{F}_{c^+(1)}) = \sum_{n \gg -\infty} c_{c^+(1)}(n) q^n$.

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Proof.

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1[f_\beta] + t_2[f_{\beta'}] \in \mathbb{H}^1_{\mathrm{par}}(X_{K_p}, \nabla_{k-2})_f$$

$$D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta + t_2 f_{\beta'} + D^{k-1} h$$

Apply U^w as $w \to \infty$. □

Moral of the story

Mock modular forms and harmonic Maass forms can be approached cohomologically, both from a complex and p-adic point of view.