Quantum Mock Modular Forms Arising From eta-theta Functions

Holly Swisher

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Joint with Amanda Folsom, Sharon Garthwaite, Soon-Yi Kang, Stephanie Treneer (AIM SQuaRE) and Brian Diaz, Erin Ellefsen (NSF REU)

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Mock Modular Forms

- Ramanujan's last letter to Hardy indicated a fascination wth what he called mock theta functions, whose modular properties remained unknown for nearly a century.
- Zwegers showed that these (holomorphic) mock theta functions could be completed to form a harmonic Maass form, which is modular but no longer holomorphic.
- Brunier and Funke defined these harmonic Maass forms, which in addition to being modular, also satisfy bounded growth conditions and are annihilated by a weighted Laplacian.
- Zagier defined *mock modular form* to mean a holomorphic part of a harmonic Maass form.

Zwegers and shadows

For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C}\backslash(\mathbb{Z}\tau + \mathbb{Z})$, Zwegers defines

$$\mu(u, v; \tau) := \frac{e^{\pi i u}}{\vartheta(v;\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi i n v} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i u} q^n},$$

where $\vartheta(v;\tau)$ is the weight 1/2 theta function given by

$$\vartheta(v;\tau) := \sum_{n \in \mathbb{Z}} e^{2\pi i (n+\frac{1}{2})(v+\frac{1}{2})} q^{\frac{1}{2}(n+\frac{1}{2})^2}.$$

Zwegers shows that $\mu(u, v; \tau)$ can be completed to form a non-holomorphic, two-variable modular Jacobi form $\widehat{\mu}(u, v; \tau)$ of weight $1/2$.

Current landscape of mmf research

Since the fundamental work of Zwegers, there has been an incredible blossoming of new work on mock modular forms, including connections to

- Partition Theory (Garvan, Jennings-Shaffer, Bringmann, Ono, and others)
- Geometry and p-adic analysis (Candelori, Castella)
- Moonshine (Duncan, Griffin, Ono)
- Plus much more!

Shadows

- A harmonic Maass form $\widehat{f}$ of weight κ is mapped to a classical modular form of weight $2-\kappa$ by the differential operator $\xi_\kappa := 2iy^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}}$
- The image of $\widehat{f}$ under ξ_κ is called the *shadow* of f.
- Our work was motivated by an interest in relating shadows and mock modular forms.

Work of Zwegers

For appropriate choices of u, v, a, b the Jacobi form $\widehat{\mu}(u, v; \tau)$ has shadow related to the unary theta function defined for $\tau \in \mathbb{H}$ by

$$g_{a,b}(\tau) := \sum_{n\in\mathbb{Z}} (n+a) e^{2\pi i b(n+a)} q^{\frac{(n+a)^2}{2}}.$$

In particular, when a and b are rational $g_{a,b}$ is a modular form of weight $3/2$.

Work of Lemke Oliver

Lemke Oliver establishes which eta-quotients of weight 1/2 and 3/2 are theta functions (or linear combinations). Weight 3/2:

$$\begin{aligned}
E_1(\tau) &= \eta(8\tau)^3 &&= \sum_{n\geq 1}\left(\frac{-4}{n}\right) nq^{n^2},\\
E_2(\tau) &= \frac{\eta(16\tau)^9}{\eta(8\tau)^3\eta(32\tau)^3} &&= \sum_{n\geq 1}\left(\frac{-2}{n}\right) nq^{n^2},\\
E_3(\tau) &= \frac{\eta(3\tau)^2\eta(12\tau)^2}{\eta(6\tau)} &&= \sum_{n\geq 1}\left(\frac{n}{3}\right) nq^{n^2},\\
E_4(\tau) &= \frac{\eta(48\tau)^{13}}{\eta(24\tau)^5\eta(96\tau)^5} &&= \sum_{n\geq 1}\left(\frac{-6}{n}\right) nq^{n^2},\\
E_5(\tau) &= \frac{\eta(24\tau)^5}{\eta(48\tau)^2} &&= \sum_{n\geq 1}\left(\frac{n}{12}\right) nq^{n^2},\\
E_6(\tau) &= \frac{\eta(6\tau)^5}{\eta(3\tau)^2} &&= \sum_{n\geq 1}\left(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right)\right) nq^{n^2}.
\end{aligned}$$

Constructing Examples

For $\tau \in \mathbb{H}$, we define for $a, b \in \mathbb{R}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$ the function

$$\widehat{M}_{a,b}(\tau) := -\sqrt{2} e^{2\pi i a \left(b+\frac{1}{2}\right)} q^{-\frac{a^2}{2}} \widehat{\mu}(u, v; \tau). \tag{1}$$

Proposition

Let $\tau \in \mathbb{H}$, and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$. If $u - v = a\tau - b$ for some $a, b \in \mathbb{R}$, then the function $\widehat{M}_{a,b}(\tau)$ satisfies

(i) $\xi_{\frac{1}{2}} \left(\widehat{M}_{a,b}(\tau) \right) = g^{c}_{a+\frac{1}{2}, b+\frac{1}{2}}(\tau)$,

(ii) $\Delta_{\frac{1}{2}} (\widehat{M}_{a,b}(\tau)) = 0$.

We denote the holomorphic part of $\widehat{M}_{a,b}$ by $M_{a,b}$:

$$M_{a,b}(\tau) := -\sqrt{2} e^{2\pi i a \left(b+\frac{1}{2}\right)} q^{-\frac{a^2}{2}} \mu(u, v; \tau).$$

Constructing Examples

- We can express each weight 3/2 eta-theta E_m in term of Zwegers' $g_{a,b}$ functions. For example, $E_1(\tau) = 4g_{\frac{1}{4},0}(32\tau)$.
- We can express each weight 1/2 eta-theta e_n in terms of $\vartheta(v;\tau)$. For example, $\vartheta(\frac{\tau}{2};\tau) = -iq^{\frac{1}{8}}e_1(\frac{\tau}{2})$.
- Thus for each pair E_m, e_n (up to a few exceptions) we can construct a holomorphic function $V_{mn} = \star\mu(u,v;\tau)$ that includes the factor e_n such that $u - v = a\tau - b$ so that the shadow is related to E_m (up to normalizations).

Mock Modular Result

Theorem (Folsom, Garthwaite, Kang, S-, Treneer)

The functions V_{mn} *are mock modular forms of weight* $1/2$ *with respect to the congruence subgroups* A_{mn}. *Moreover, the shadow of* V_{mn} *is given by a constant multiple of the odd eta-theta function* $E_m\left(\frac{2\tau}{c_m^2}\right)$. *In particular, the functions* V_{mn} *may be completed to form harmonic Maass forms* $\widehat{V}_{mn}$ *of weight* $1/2$ *on* A_{mn}, *which satisfy for all* $\gamma_{mn} = \left(\begin{smallmatrix} a_{mn} & b_{mn} \\ c_{mn} & d_{mn} \end{smallmatrix}\right) \in A_{mn}$, *and* $\tau \in \mathbb{H}$,

$$\widehat{V}_{mn}(\gamma_{mn}\tau) = \star\nu(\gamma_{mn})^{-3}(c_{mn}\tau + d_{mn})^{\frac{1}{2}}\widehat{V}_{mn}(\tau),$$

where $\star$ *is an explicit root of unity, and* $\nu(\gamma_{mn})$ *is the usual eta-function multiplier.*

Table for $m = 1$

Table: Mock theta functions with normalized shadow $E_1(\tau)$.

$V_{1n}(\tau)$	Series	$w_1 q^{t_1} \mu\left(u_\tau^{(1n)}, v_\tau^{(1n)}; \tau\right)$
$V_{11}(\tau)$	$\frac{q^{-9/32}}{e_1(\tau/2)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(n+1)^2/2}}{1+q^{n+1/4}}$	$-q^{-1/32}\mu(\frac{\tau}{4}+\frac{1}{2}, \frac{\tau}{2}; \tau)$
$V_{12}(\tau)$	$\frac{-q^{-9/32}}{e_2(\tau/2)} \sum_{n\in\mathbb{Z}} \frac{q^{(n+1)^2/2}}{1-q^{n+1/4}}$	$-q^{-1/32}\mu(\frac{\tau}{4}, \frac{\tau}{2}-\frac{1}{2}; \tau)$
$V_{13}(\tau)$	$\frac{q^{-9/32}}{e_3(\tau/72)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(n+5/6)^2/2}}{1+q^{n+1/12}}$	$-q^{-1/32}\mu(\frac{\tau}{12}+\frac{1}{2}, \frac{\tau}{3}; \tau)$
$V_{14}(\tau)$	$\frac{-q^{-9/32}}{e_4(\tau/72)} \sum_{n\in\mathbb{Z}} \frac{q^{(n+5/6)^2/2}}{1-q^{n+1/12}}$	$-q^{-1/32}\mu(\frac{\tau}{12}, \frac{\tau}{3}-\frac{1}{2}; \tau)$
$V_{15}(\tau)$	$\frac{q^{-9/32}}{e_5(\tau/32)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(n+3/4)^2/2}}{1+q^{n}}$	$-q^{-1/32}\mu(\frac{1}{2}, \frac{\tau}{4}; \tau)$
$V_{16}(\tau)$	—	$-q^{-1/32}\mu(0, \frac{\tau}{4}-\frac{1}{2}; \tau)$

Comparison with known Mock Theta Functions

$$-q^{1/24}V_{41}(12\tau) = \psi(q), \quad q^{1/24}V_{58}(3\tau) = \chi(q),$$
$$q^{-2/3}V_{64}(6\tau) = \rho(q), \quad -q^{1/8}V_{21}(4\tau) = A(q),$$
$$-q^{1/8}V_{12}(4\tau) = U_1(q), \quad 2q^{1/8}V_{15}(4\tau) = U_0(q),$$

where $\psi(q)$, $\chi(q)$, and $\rho(q)$ are Ramanujan's third order mock thetas, $A(q)$ is Ramanujan's second order mock theta, and $U_1(q)$ and $U_0(q)$ are Gordon and McIntosh's eighth order mock thetas.

Quantum Sets

We call a subset $S \subseteq \mathbb{Q}$ a *quantum set* for a function F with respect to the group $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ if both $F(x)$ and $F(Mx)$ exist (are non-singular) for all $x \in S$ and $M \in G$.

Quantum Modular Forms

For $k \in \frac{1}{2}\mathbb{Z}$, a *quantum modular form of weight k on the set S for the group G* is a complex-valued function f such that S is a quantum set for f with respect to the group $G \subseteq \mathrm{SL}_2(\mathbb{Z})$, and for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, and for all $x \in S$ ($x \neq -\frac{d}{c}$), the functions

$$h_{f,\gamma}(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

are suitably continuous or analytic in (a subset of) $\mathbb{R}$. We will take this to mean real analytic.
(The $\epsilon(\gamma)$ are appropriate complex numbers, like those in the theory of half-integer weight modular forms.)

Current landscape of qmf research

- Zagier first defined quantum modular forms in 2010 and demonstrated some interesting examples
- Quantum modular forms have naturally arisen in several places in work by Andrews, Dyson, Hickerson; Bryson, Ono, Pitman, Rhoades; Bringmann, Creutzig, Rolen; Folsom, Ono, Rhoades; as well as others.
- They arise, for example, in the study of Kontsevich's"strange" function, unimodal sequences, and false theta functions.

Quantum Modular Result

Theorem (Folsom, Garthwaite, Kang, S-, Treneer)

The functions V_{mn} are quantum modular forms of weight $1/2$ on the sets S_{mn} for the groups G_{mn}. In particular,

(i) *For all $x \in \mathbb{H} \cup S_{mn} \setminus \left\{\frac{-1}{2}\right\}$,*

$$V_{mn}(x) + \zeta_4^{\ell_m}(2x+1)^{-\frac{1}{2}} V_{mn}\left(M_2 x\right) = -\frac{i}{c_m} \int_{\frac{1}{2}}^{i\infty} \frac{E_m\left(\frac{2u}{c_m^2}\right)}{\sqrt{-i(u+x)}}\, du.$$

(ii) *For all $x \in \mathbb{H} \cup S_{mn}$,*

$$V_{mn}(x) - \zeta_{a_m}^{\kappa_{mn}} V_{mn}(x + \kappa_{mn} b_m) = 0.$$

Curious Application

We obtain interesting corollaries which give closed expressions for the Eichler integrals of the eta-theta functions E_m in terms of the following truncated q-hypergeometric series

$$F_{h,k}(z_1, z_2) := \sum_{n=0}^{k-1} \frac{(-\zeta_{2k}^h; \zeta_{2k}^h)_n \zeta_{4k}^{n(n+1)h}}{(z_1; \zeta_{2k}^h)_{n+1} (z_2; \zeta_{2k}^h)_{n+1}}.$$

A particularly pretty result is that for $m \in \{1, 2, 5\}$,

$$F_{h,k}(-i^{\ell_m - 3} \zeta_{c_m k}^h, -i^{3-\ell_m} \zeta_{a_m k}^{d_m h}) + F_{h,k}(i^{\ell_m - 3} \zeta_{c_m k}^h, i^{3-\ell_m} \zeta_{a_m k}^{d_m h}) = 0.$$

Example of Corollaries

The Eichler integral of the eta-quotient E_1 may be evaluated as

$$\frac{-i}{8}\int_{1/2}^{i\infty}\frac{E_1(z/32)dz}{\sqrt{-i(z+\frac{1}{3})}} = \zeta_{32}^{-7}\sum_{n=0}^{2}\frac{(-\zeta_6;\zeta_6)_n\zeta_{12}^{n(n+1)}}{(i\zeta_{24};\zeta_6)_{n+1}(-i\zeta_8;\zeta_6)_{n+1}}$$

$$-\left(\tfrac{3}{5}\right)^{\frac{1}{2}}\zeta_{160}^{-37}\sum_{n=0}^{4}\frac{(-\zeta_{10};\zeta_{10})_n\zeta_{20}^{n(n+1)}}{(i\zeta_{40};\zeta_{10})_{n+1}(-i\zeta_{40}^{3};\zeta_{10})_{n+1}}$$

$$\approx .05461 + .00825i.$$

Example of Corollaries

We also have the following curious algebraic identity.

$$\sum_{n=0}^{2} \frac{(-\zeta_6;\zeta_6)_n \zeta_{12}^{n(n+1)}}{(i\zeta_{24};\zeta_6)_{n+1}(-i\zeta_8;\zeta_6)_{n+1}} + \sum_{n=0}^{2} \frac{(-\zeta_6;\zeta_6)_n \zeta_{12}^{n(n+1)}}{(-i\zeta_{24};\zeta_6)_{n+1}(i\zeta_8;\zeta_6)_{n+1}} = 0.$$

While the above may appear elementary, term by term the two sums appearing are quite different.

Lower Half Plane

It is natural to ask if the functions V_{mn} also extend into the lower half-plane $\mathbb{H}^- := \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$.
For $z \in \mathbb{H}^-$, define

$$\widetilde{E}_1(z) = \sum_{n\geq 1} \left(\frac{-4}{n}\right) e^{-2\pi i z n^2}, \quad \widetilde{E}_4(z) = \sum_{n\geq 1} \left(\frac{-6}{n}\right) e^{-2\pi i z n^2},$$

$$\widetilde{E}_2(z) = \sum_{n\geq 1} \left(\frac{-2}{n}\right) e^{-2\pi i z n^2}, \quad \widetilde{E}_5(z) = \sum_{n\geq 1} \left(\frac{n}{12}\right) e^{-2\pi i z n^2},$$

$$\widetilde{E}_3(z) = \sum_{n\geq 1} \left(\frac{n}{3}\right) e^{-2\pi i z n^2}, \quad \widetilde{E}_6(z) = \sum_{n\geq 1} \left(2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right)\right) e^{-2\pi i z n^2}.$$

Lower Half Plane

Proposition

The functions $\widetilde{E}_m$ are quantum modular forms of weight $1/2$. In particular, for any $x \in S_{mn}$, up to multiplication by a constant, the functions $\widetilde{E}_m\left(-2x/c_m^2\right)$ satisfy the transformation laws given in the quantum modularity theorem for the functions $V_{mn}(x)$.

Outline of Proof when $n = 1$

Theorem (Kang)

If $\alpha \in \mathbb{C}$ such that $\alpha \notin \frac{1}{2}\mathbb{Z}\tau + \frac{1}{2}\mathbb{Z}$, then

$$\mu\left(2\alpha, \frac{\tau}{2}; \tau\right) = iq^{\frac{1}{8}} g_2(e(\alpha); q^{\frac{1}{2}}) - e(-\alpha)q^{\frac{1}{8}} \frac{\eta(\tau)^4}{\eta(\frac{\tau}{2})^2 \vartheta(2\alpha; \tau)},$$

where g_2 is the universal mock theta function defined by

$$g_2(z; q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{n(n+1)/2}}{(z; q)_{n+1}(z^{-1}q; q)_{n+1}}.$$

Outline of Proof when $n = 1$

- Using Kang's theorem, determine a quantum set S_{m1} and group G_{m1} for which V_{m1} is well-defined for all $x \in S_{m1}$, and Mx for any $M \in G_{m1}$.
- Use Zwegers' transformations for μ on $\mathbb{H}$ to obtain explicit transformation properties for V_{m1} on $\mathbb{H}$ with respect to G_{m1}. A couple of error terms will arise.
- Using Zwegers, convert the error terms to Eichler integrals and see them beautifully reduce to a single integral (when set up properly).
- Put everything together to obtain explicit results in quantum modularity theorem.

Definitions for Quantum Sets

$$
\begin{aligned}
S &:= \{h/k \in \mathbb{Q} \,|\, h \in \mathbb{Z}, k \in \mathbb{N}, \gcd(h,k) = 1, h \equiv 1 \pmod 2\} \\
S' &:= \{h/k \in S \mid h \equiv \pm 1 \pmod 6\} \\
S_{ev} &:= \{h/k \in S \mid k \equiv 0 \pmod 2\} \\
S_{od} &:= \{h/k \in S \mid k \equiv 1 \pmod 2\}
\end{aligned}
$$

Quantum Sets

$$\begin{aligned}
S_{11}, S_{17}, S_{21}, S_{27}, S_{41}, S_{45}, S_{47} &:= S \\
S_{12}, S_{18}, S_{22}, S_{28}, S_{42}, S_{52} &:= S_{ev} \\
S_{13}, S_{23}, S_{31}, S_{34}, S_{35}, S_{36}, S_{43}, S_{53}, S_{61}, S_{63}, S_{64}, S_{65}, S_{66} &:= S' \\
S_{14}, S_{15}, S_{24}, S_{26} &:= S_{od} \\
S_{32}, S_{33}, S_{62} &:= S' \cap S_{ev} \\
S_{37}, S_{68} &:= S' \cap S_{od} \\
S_{44} &:= S' \cup S_{od} \\
S_{46}, S_{48}, S_{51}, S_{55}, S_{56}, S_{57}, S_{58} &:= S' \cup S_{ev}.
\end{aligned}$$

Groups for Quantum Modularity

$$G_{12}, G_{15}, G_{22}, G_{26} := \langle \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right) \rangle \subset \Gamma_0(2) \cap \Gamma^0(4),$$

$$\begin{matrix} G_{13}, G_{14}, G_{17}, G_{18}, G_{23}, \\ G_{24}, G_{27}, G_{28}, G_{35}, G_{36}, \\ G_{4n}, G_{55}, G_{56}, G_{65}, G_{66} \end{matrix} := \langle \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 12 \\ 0 & 1 \end{smallmatrix}\right) \rangle \subset \Gamma_0(2) \cap \Gamma^0(12),$$

$$\begin{matrix} G_{32}, G_{33}, G_{34}, G_{37}, \\ G_{52}, G_{53}, G_{57}, G_{58}, \\ G_{62}, G_{63}, G_{64}, G_{68} \end{matrix} := \langle \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 6 \\ 0 & 1 \end{smallmatrix}\right) \rangle \subset \Gamma_0(2) \cap \Gamma^0(6).$$

Question for REU Students

- Are there patterns here for the quantum sets and groups appearing?
- How special is it that these V_{mn} functions are quantum modular?
- Could one prove a more general result that captures the quantum modularity properties shown for this catalog?

REU Definition of Quantum Sets

Define

$$S := \left\{ \frac{h}{k} \in \mathbb{Q} \mid h \in \mathbb{Z}, k \in \mathbb{N}, \gcd(h,k) = 1, \text{ and } h \text{ odd} \right\}.$$

Given

$$\alpha = \frac{A}{2C}\tau + \frac{a}{4},$$

where $0 \leq a \leq 3$, $\gcd(A, C) = 1$, and $0 < \frac{A}{C} < 1$, we define

$$S_\alpha = \begin{cases} \left\{ \frac{h}{k} \in S \mid C \nmid h \right\} & \text{if } a \text{ even }, \\ \left\{ \frac{h}{k} \in S \mid C \nmid 2h \right\} \cup \left\{ \frac{h}{k} \in S \mid C \nmid h \text{ and } k \text{ even} \right\} & \text{if } a \text{ odd.} \end{cases}$$

REU Definition of Quantum Groups

Recall $\alpha = \frac{A}{2C}\tau + \frac{a}{4}$. We define

$$G_\alpha = \begin{cases} \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a \text{ and } C \text{ even} \\ \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a \text{ odd and } C \text{ even} \\ \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a \text{ even and } C \text{ odd} \\ \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2C \\ 0 & 1 \end{pmatrix} \right\rangle, & \text{if } a \text{ and } C \text{ odd.} \end{cases}$$

REU Results

Theorem (Diaz, Ellefsen, S-)

Let $\alpha = \frac{A}{2C}\tau + \frac{a}{4}$, as before. Then the functions

$$V_\alpha(\tau) := i^{a+1} q^{-\frac{(2A-C)^2}{8C^2}} \mu\left(2\alpha, \frac{\tau}{2}; \tau\right).$$

are quantum modular forms on the sets S_α for the groups G_α, with explicit transformations that hold on all of $\mathbb{H} \cup S_\alpha$.

REU Results

For example, when a is odd, we have that for all $\tau \in \mathbb{H} \cup S_\alpha$,

$$V_\alpha(\tau) - (2\tau+1)^{-\frac{1}{2}} V_\alpha(M_2\tau) = \frac{-i}{2} \int_{\frac{1}{2}}^{i\infty} \frac{g_{\frac{A}{C},0}(z)}{\sqrt{-i(z+\tau)}} dz,$$

where $M_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.

Further Work and Open Questions

- It remains to obtain explicit transformations for V_α and show it is mock modular; this is work in progress.
- It may be possible to do a similar shifting process to obtain results for a wider scope of functions which encompass the entire table from Folsom, et al.
- Further generalization seems possible perhaps sacrificing some optimization of quantum sets and groups.
- A truly general result seems out of reach due to an expansion of cases based on divisibility.

Thank you!