A FEW WORDS ABOUT HIGHER WEIGHT:

WE TAKE

$$\mathcal{M}_k := \mathbb{Q}[x,y]_{k-2} \otimes_{\mathbb{Q}} \mathcal{M}_2$$

HOMOGENEOUS POLYNOMIALS IN x,y
DEGREE $k-2$

$$GL_2(\mathbb{Q}) \circlearrowright \quad (gP)(x,y) = P\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= P(dx-by, -cx+ay)$$

SO

$$GL_2(\mathbb{Q}) \circlearrowright \mathcal{M}_k$$

$$g(P \otimes \{\alpha,\beta\}) = gP \otimes \{g\alpha, g\beta\}.$$

LET $\mathcal{M}_k(\Gamma) := \mathcal{M}_k / (P\otimes\{\alpha,\beta\} - g(P\otimes\{\alpha,\beta\}))$

AND $\mathcal{B}_k := \mathbb{Q}[x,y]_{k-2} \otimes \mathcal{B}_2$

THEN $\mathcal{B}_k(\Gamma) := \mathcal{B}_k / (x - gx)$

$$\partial : \mathcal{M}_k(\Gamma) \longrightarrow \mathcal{B}_2(\Gamma)$$

$$\partial(P\otimes\{\alpha,\beta\}) = P\otimes\beta - P\otimes\alpha$$

$S_k(\Gamma) := \ker \partial.$

$$\left(S_k(\Gamma) \oplus \bar{S}_k(\Gamma)\right) \times \mathcal{S}_k(\Gamma) \to \mathbb{C}$$

$$\left\langle (f, g), P \otimes \{\alpha, \beta\} \right\rangle$$

$$:= \int_\alpha^\beta f(z)\, P(z, 1)\, dz + \int_\alpha^\beta g\, P(\bar{z}, 1)\, d\bar{z}$$

IS NONDEGENERATE, ...

[GET L-VALUES FROM THIS PAIRING, ...]

LECTURE 4: SHIMURA CURVES

(UCONN, THU, 11 AUG 2016)

BASIC IDEA: REPLACE $M_2(\mathbb{Q}) \rightsquigarrow$ QUATERNION ALGEBRA!

WARMUP: $B = \left(\frac{a,b}{\mathbb{Q}}\right)$, $a,b \in \mathbb{Z}$

$$\begin{array}{l} \;\cup\!\!| \\ \mathcal{O} = \mathbb{Z}\langle i,j\rangle = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij \end{array}$$

$$\alpha \in \mathcal{O}^\times \Leftrightarrow \mathrm{nrd}(\alpha) = \pm 1.$$

$$\alpha = t + xi + yj + zij \qquad \mathrm{nrd}(\alpha) = t^2 - ax^2 - by^2 + abz^2$$

IF $a, b < 0$, THEN $\mathcal{O}^\times$ IS FINITE, AND POSSIBLE GROUPS CAN BE CLASSIFIED.

ELSE, "NONCOMMUTATIVE PELL'S EQUATION", $\#\mathcal{O}^\times = \infty$.

LET F BE A TOTALLY REAL FIELD, $n=[F:\mathbb{Q}]$

$\mathrm{Pl}(F)$ SET OF PLACES,
$\mathrm{Pl}_\infty(F)$ REAL PLACES
$R=\mathbb{Z}_F$ RING OF INTEGERS.

THM (DIRICHLET): $R^\times \simeq \mathbb{Z}/w\mathbb{Z} \oplus \mathbb{Z}^{n-1}$
[PROOF: EMBED AS LATTICE...]

LET B BE A QUATALG / F

$\mathcal{O} \subseteq$ AN R-ORDER IN B, $\mathrm{Ram}(B) \subseteq \mathrm{Pl}(F)$ SET OF RAMIFIED PLACES

[SUBRING, F.G. AS R-MODULE, CONTAINING A BASIS FOR B.]

LET $\mathcal{O}^1 = \{\gamma \in \mathcal{O} : \mathrm{nrd}(\gamma)=1\} \subseteq B^1$

<u>EX</u>: $B = M_2(F)$

$\mathcal{O} = M_2(R)$

$\mathcal{O}^\times = GL_2(R)$

$\mathcal{O}^1 = SL_2(R)$.

$$\underset{\substack{\wr\wr \\ \mathbb{Z}^n}}{R} \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v \in \mathrm{Pl}_\infty(F)} \mathbb{R} \simeq \mathbb{R}^n \text{ IS DISCRETE.}$$

$$\Rightarrow \underset{\text{DISCRETE}}{\mathcal{O} \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{R}} \simeq \prod_{v \in \mathrm{Pl}_\infty(F)} B_v \text{ IS DISCRETE}$$

EACH $B_v \simeq \begin{cases} \mathbb{H}, & v \text{ RAM} \\ M_2(\mathbb{R}), & v \text{ SPLIT} \end{cases}$

LET $\Omega = \mathrm{Ram}(B) \cap \mathrm{Pl}_\infty(F)$

$$\Rightarrow \mathcal{O}^1 \hookrightarrow \prod_{v \in \mathrm{Pl}_\infty(F)} B_v^1$$

$B_v^1 \simeq \begin{cases} \mathbb{H}^1, & v \in \Omega \\ SL_2(\mathbb{R}), & v \notin \Omega. \end{cases}$

COMPACT $\Leftrightarrow$ $v \in \Omega$.

THM: $\mathcal{O}^1$ IS FINITE $\Leftrightarrow$ $Pl_\infty(F) \subseteq \Omega$

I.E. ALL PLACES ARE RAMIFIED.

$$\mathcal{O}^1 \hookrightarrow \prod_{v \in Pl_\infty(F) \setminus Ram(B)} B_v^1 \quad \text{IS DISCRETE}$$

[PROJECT AWAY FROM COMPACT FACTORS. $H \leq G_1 \times G_2$, G_1 COMPACT, H DISCRETE IN PRODUCT $\Rightarrow$ $H \leq G_2$ IS DISCRETE.]

[NO FURTHER PROJECTION IS DISCRETE.]

TO GET SOMETHING WHICH IS IN JUST 1 $SL_2(\mathbb{R})$, SUPPOSE $\#\Omega = n-1$ [I.E. ALL BUT ONE PLACE IS RAMIFIED $\Leftrightarrow$ EXACTLY ONE SPLIT.]

THEN $\mathcal{O}^1/\{\pm 1\} \hookrightarrow PSL_2(\mathbb{R}) \circlearrowright \mathcal{H}^2$.

NICE FUCHSIAN GROUP

[CALL GROUPS COMMENSURABLE WITH SUCH A $\Gamma(\mathcal{O})$ ARITHMETIC.]

4/11

EX: LET $F = \mathbb{Q}(\sqrt{29})$.

$R = \mathbb{Z}_F = \mathbb{Z}[w]$, $w = \dfrac{1+\sqrt{29}}{2}$. $d_F = 29$

$h^+(F) = h(F) = 1$, $u = w+2$ IS A FUNDAMENTAL UNIT.

↑ NARROW CLASS NUMBER

LET $B = \left(\dfrac{-1, u}{F}\right)$, $i^2 = -1$, $j^2 = u$, $ji = -ij$.

$\mathrm{Ram}(B) = \{(2), \infty_2\}$, so $\operatorname{disc} B = \mathfrak{D} = (2)$

$\infty_2 : w \mapsto -\sqrt{29}$.

∞_1 IS ~~UNRAMIFIED~~ SPLIT, SO GIVES

$$\iota_\infty : B \hookrightarrow B \otimes_F \mathbb{R} \simeq M_2(\mathbb{R})$$

$$i, j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{u} & 0 \\ 0 & -\sqrt{u} \end{pmatrix} = \begin{pmatrix} 2.27\ldots & 0 \\ 0 & -2.27\ldots \end{pmatrix}$$

LET $\mathcal{O} := R + Ri + Rj + Rk$

$k := \frac{1}{2}(1+i)(w+1+j)$.

[$\mathcal{O}$ IS A MAXIMAL R-ORDER.]

5/11

LET $\Gamma = \iota_\infty(\mathcal{O}^1/\{\pm 1\}) \subset PSL_2(\mathbb{R})$.

$X := X(\Gamma) = \Gamma \backslash \mathfrak{H}^2$ IS A RIEMANN SURFACE (SHIMURA CURVE) ASSOCIATED TO Γ.

[LOOK MA, NO CUSPS!].

A FEW BASIC FORMULAS.

$$A = \operatorname{area}(X) = \frac{4}{(2\pi)^{2n}} d_F^{3/2} \zeta_F(2)\, \overbrace{\Phi(\mathfrak{D})}^{\prod_{\mathfrak{p} \mid \mathfrak{D}} (N(\mathfrak{p})-1)}$$

$$= \frac{4}{(2\pi)^4} (\sqrt{29})^{3/2} \zeta_F(2)(4-1) = \frac{3}{2}.$$

RIEMANN-HURWITZ:

$$A = 2g(X) - 2 + \sum_q e_q \left(1 - \frac{1}{q}\right)$$

$\uparrow$ # ELLIPTIC CYCLES OF ORDER $q \in \mathbb{Z}_{\geq 2}$ IN Γ

$e_2 = 3, \quad e_q = 0 \quad q > 2$

$2g - 2 = 0 \Rightarrow g = 1.$ SAY Γ HAS SIGNATURE $(1; 2, 2, 2)$.

[MAKE A SPECIAL KIND OF FUNDAMENTAL DOMAIN.]

LET $z_0 \in \mathcal{H}^2$ HAVE

$$\mathrm{Stab}_\Gamma(z_0) = \{ \gamma \in \Gamma : \gamma z_0 = z_0 \} = \{1\}.$$

LET

$$\Omega(z_0) = \{ z \in \mathfrak{h} : \underset{\substack{\uparrow \\ \text{HYPERBOLIC DISTANCE}}}{d}(z, z_0) \leq d(\gamma z, z_0) \ \forall \gamma \in \Gamma \}.$$

$\Omega(z_0)$ IS THE <u>DIRICHLET DOMAIN</u> FOR Γ AT z_0.

IT IS: A FUNDAMENTAL DOMAIN

CONVEX, HYPERBOLIC POLYGON

(CLOSED, CONNECTED)

BOUNDARY CONSISTS OF FINITELY MANY SIDES.

WHICH CAN BE PAIRED BY

A <u>SIDE PAIRING</u>.

THIS IS ALGORITHMIC. [SEE EXERCISES.]

[WE GET A <u>REDUCTION ALGORITHM</u>.]

[SHOW HOW WORKS WITH SLIDES...?]

WE OBTAIN PRESENTATION

$$\Gamma \xrightarrow{\sim} \langle \gamma, \gamma', \delta_1, \delta_2, \delta_3 : \delta_1^2 = \delta_2^2 = \delta_3^2 = \gamma\gamma'\gamma^{-1}\gamma'^{-1}\delta_1\delta_2\delta_3 = 1 \rangle$$

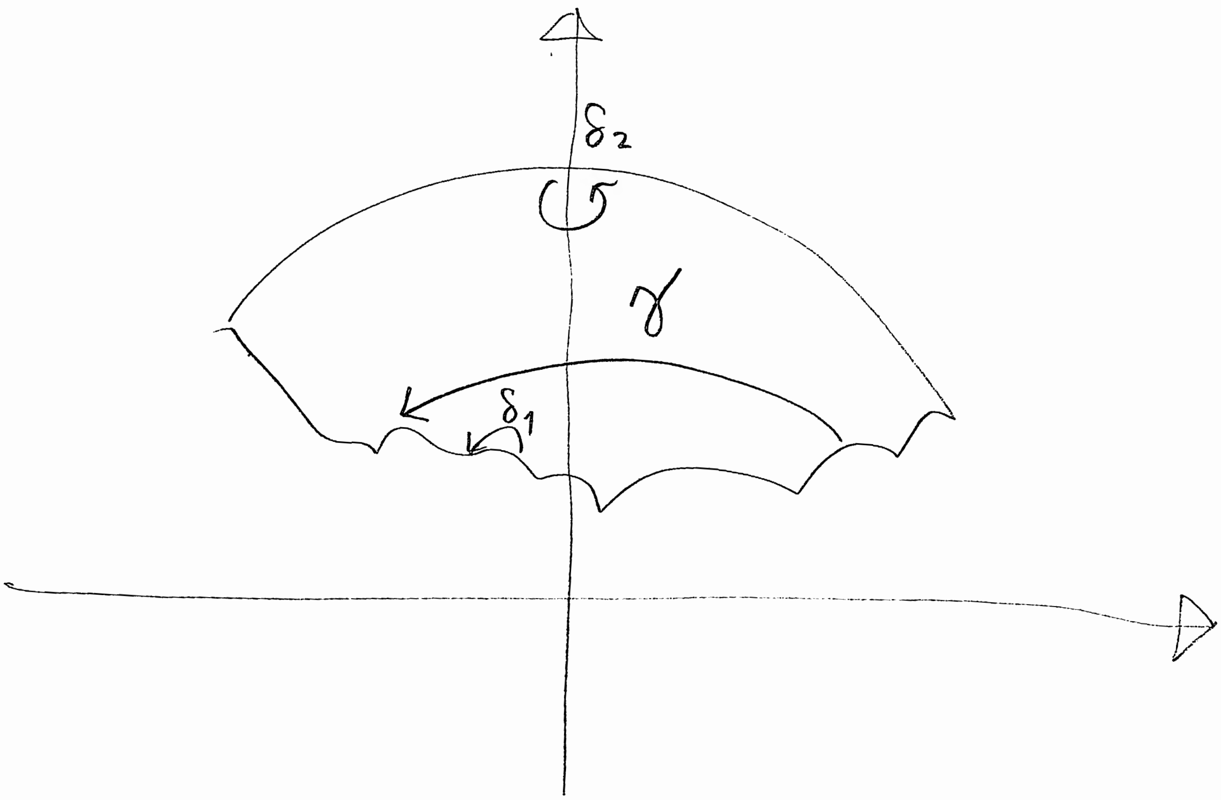
WHERE

$$\gamma = \frac{-1}{2}(w+1) + \frac{1}{2}(w+3)i - \frac{3}{2}i + \frac{1}{2}ij$$

$\vdots$

$$\delta_2 = i$$

$\vdots$



WE HAVE ISOMORPHISMS OF $\mathbb{C}$-VECTOR SPACES, COMMUTING WITH HECKE OPERATORS:

$$S_2(\Gamma) = \{ f : \mathfrak{h} \to \mathbb{C} \mid f(\gamma z)\, d(\gamma z) = f(z)\, dz \quad \forall \gamma \in \Gamma \}$$

EICHLER-SHIMURA $\cong$ (DUAL TO INTEGRATION)

$$\downarrow$$

$$H^1(\Gamma, \mathbb{C})^+$$

$$\wr\!\wr$$

$$\mathrm{Hom}(\Gamma, \mathbb{C})^+ = \mathbb{C} f_\gamma$$

WHERE $f_\gamma(\gamma) = 1$

$f_\gamma(\gamma') = 0$

$f_\gamma(\delta_i) = 0$

JACQUET-LANGLANDS $\cong$

$$\downarrow$$

$$S_2((2))^{\text{new}}$$

$$\parallel$$

SPACE OF HILBERT NEWFORMS OF PARALLEL WEIGHT 2 AND LEVEL (2).

LET $\mathcal{S}_2(\Gamma)$ BE THE $\mathbb{C}$-VECTOR SPACE GENERATED BY $\{v, \gamma v\}$ FOR γ IN A SIDE PAIRING FOR Γ AND v THE MIDPOINT OF THE SIDE PAIRED BY γ.

THEN $H_1(X, \mathbb{C}) \simeq H_1(\Gamma, \mathbb{C}) \simeq \mathcal{S}_2(\Gamma)$, DUAL TO $S_2(\Gamma)$.

CALL $\mathcal{S}_2(\Gamma)$ THE SPACE OF <u>DIRICHLET MODULAR SYMBOLS</u> FOR Γ (RELATIVE TO z_0).

HECKE OPERATORS:

LET $\mathfrak{p} = (w+1)$, $N\mathfrak{p} = p = 5$.

COMPUTE $H^1(\Gamma, \mathbb{C})^+ \circlearrowleft T_{\mathfrak{p}}$, $T_{\mathfrak{p}} f_\gamma = a_{\mathfrak{p}} f_\gamma$.

LET $\iota_{\mathfrak{p}} : \mathcal{O} \hookrightarrow M_2(R_{\mathfrak{p}}) \simeq M_2(\mathbb{Z}_5)$

$$i, j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pmod 5$$

$[j^2 = u \equiv 1 \pmod{w+1}]$

LET $I_\infty, I_0, \ldots, I_4$ REPRESENT THE LEFT IDEALS

$$\underset{(x:y)}{I_a} = \mathcal{O}_{\iota_p}^{-1}\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} + p\mathcal{O}$$

E.G. $I_0 = \mathcal{O}\xi_0 = \mathcal{O}(i-ij) + p\mathcal{O}$

$$= (-w+3) + wi + j + ij.$$

THEN BY STRONG APPROXIMATION,

$I_a = \mathcal{O}\xi_a$, $\xi_a \in \mathcal{O}$.

FOR $f: \Gamma \longrightarrow \mathbb{C}$, $\gamma \in \Gamma$,

$\exists \{\delta_a \in \Gamma\}_{a \in \mathbb{P}^1(\mathbb{F}_p)}$, $\gamma^* \in \mathrm{Sym}(\mathbb{P}^1(\mathbb{F}_p))$

S.T. $\xi_{\gamma^* a} = \delta_a \xi_a$ $\quad \forall a \in \mathbb{P}^1(\mathbb{F}_p)$.

DEFINE $(T_p f)(\gamma) = \displaystyle\sum_{a \in \mathbb{P}^1(\mathbb{F}_p)} f(\delta_a \gamma)$.

E.G. $\xi_4 \gamma \xi_0^{-1} = \gamma' \delta_2 \gamma' \delta_1 \delta_3$

$N\mathfrak{p}$	$a(\mathfrak{p})$
5	1
7	-2
9	5
13	-1
2^3	6

MATCHES: $E(1682c1): y^2 + xy = x^3 + x^2 - 51318x - 4555676$

$1682 = 2 \cdot 29^2$

TAKE OVER F.

TWIST E_F BY $-4\sqrt{29}$, TO GET E_F', GETS CONDUCTOR (2), AND

$X \sim E_F'$.

$$E_F': y^2 + (w+1)xy = x^3 + (-w+1)x^2 + (-11w-20)x + (23w+52)$$