Gauss and the Arithmetic-Geometric Mean

David A. Cox

Department of Mathematics and Statistics
Amherst College
dacox@amherst.edu

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1. Math
   - The Arithmetic-Geometric Mean
   - Elliptic Integrals
   - Complex Numbers
   - Theta Functions, Modular Functions, and Modular Forms

2. Gauss
   - Gauss's Mathematical Diary
   - A Most Important Theorem
   - Fundamental Domains
   - Letter to Bessel

The AGM

Let $a \geq b$ be positive real numbers and set

$$a_1 = \tfrac{1}{2}(a+b) \qquad \text{(arithmetic mean)}$$
$$b_1 = \sqrt{ab} \qquad \text{(geometric mean)}$$

The Arithmetic Mean-Geometric Mean Inequality

$$\tfrac{1}{2}(a+b) \geq \sqrt{ab}$$

It follows that $a_1 \geq b_1$, so we can iterate.

Example

n	a_n	b_n
0	1.4142135623730950488	1.0000000000000000000
1	1.2071067811865475244	1.1892071150027210667
2	1.1981569480946342955	1.1981235214931201226
3	1.1981402347938772090	1.1981402346773072057
4	1.1981402347355922074	1.1981402347355922074

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3	1.198140234793877209083	1.198140234677307205798
4	1.198140234735592207441	1.198140234735592207439

Theorem and Definition

If $a \geq b > 0$, define $(a_0, b_0) = (a, b), (a_{n+1}, b_{n+1}) = (\frac{1}{2}(a_n + b_n), \sqrt{a_n b_n})$.

Theorem

1. $a = a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq b_n \geq \cdots \geq b_2 \geq b_1 \geq b_0 = b$.
2. $a_n - b_n \leq 2^{-n}(a - b)$.
3. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Definition

The arithmetic-geometric mean (agM for short) of $a \geq b > 0$ is

$$M(a, b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Examples

$$M(a, a) = a \quad \text{for any } a > 0$$

$$M(\sqrt{2}, 1) = 1.1981402347355922074\ldots$$

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An Integral Formula

Theorem

$$M(a,b)\int_0^{\pi/2}\frac{d\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\frac{\pi}{2}.$$

Proof.

Let $I(a,b)$ denote the integral and introduce a new variable ϕ' such that

$$\sin\phi = 2a\sin\phi'/(a+b+(a-b)\sin^2\phi').$$

"After the development has been made correctly, it will be seen" that

$$(a^2\cos^2\phi+b^2\sin^2\phi)^{-1/2}\,d\phi=(a_1^2\cos^2\phi'+b_1^2\sin^2\phi')^{-1/2}\,d\phi'$$

If we set $\mu = M(a,b)$, it follows that

$$I(a,b)=I(a_1,b_1)=\cdots=I(\mu,\mu)=\int_0^{\pi/2}\frac{d\phi}{\sqrt{\mu^2\cos^2\phi+\mu^2\sin^2\phi}}=\frac{\pi}{2\mu}.$$ □

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An Elliptic Integral

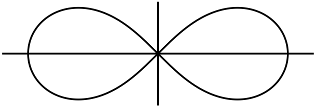
Setting $z = \cos\phi$ gives the elliptic integral

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \int_0^1 \frac{dz}{\sqrt{(b^2+(a^2-b^2)z^2)(1-z^2)}}$$

corresponding to the elliptic curve $w^2 = (b^2+(a^2-b^2)z^2)(1-z^2)$.
The very first elliptic integral, discovered by Bernoulli in 1691, is

$$\int_0^{\pi/2} \frac{d\phi}{\sqrt{2\cos^2\phi + \sin^2\phi}} = \int_0^1 \frac{dz}{\sqrt{1-z^4}}.$$

This integral is denoted $\varpi/2$ and equals the first quadrant arc length of the lemniscate $r^2 = \cos(2\theta)$.



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$$M(\sqrt{2},1) = \frac{\pi}{\varpi}$$

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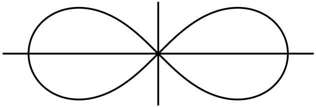

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Application

Properties of the agM

$$M(\lambda a, \lambda b) = \lambda M(a, b)$$

$$M(a+b, a-b) = M(a, \sqrt{a^2 - b^2}) = M(a, c), \quad c = \sqrt{a^2 - b^2}.$$

Take agM sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ with $a_0 = 1, b_0 = 1/\sqrt{2}$. Define $c_n = \sqrt{a_n^2 - b_n^2}$. Using properties of elliptic integrals, one can show:

Theorem

$$\pi = \frac{4M(1, 1/\sqrt{2})^2}{1 - \sum_{n=1}^{\infty} 2^{n+1} c_n^2} = \frac{2M(\sqrt{2}, 1)^2}{1 - \sum_{n=1}^{\infty} 2^{n+1} c_n^2}.$$

This formula and its variants have been used to compute π to 1,000,000,000 decimal places.

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The Complex agM

When $a, b \in \mathbb{C}$, one can still form the agM sequences

$$(a_0, b_0) = (a, b),\ (a_{n+1}, b_{n+1}) = (\frac{1}{2}(a_n + b_n), \sqrt{a_n b_n}),$$

except that at each stage, we get two choices for the square root. This leads to uncountably many possible sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$. But some of them are nice.

Example

n	a_n	b_n
0	3.0000000	1.0000000
1	2.0000000	-1.7320508
2	0.1339746	$1.8612098i$
3	$0.0669873 + 0.9306049i$	$0.3530969 + 0.3530969i$
4	$0.2100421 + 0.6418509i$	$0.2836930 + 0.6208239i$
5	$0.2468676 + 0.6313374i$	$0.2470649 + 0.6324002i$
5	$0.24699625 + 0.6318688i$	$0.24699625 + 0.6318685i$

Right Choices and Good Sequences

We will assume $a, b \in \mathbb{C}$ with $a, b \neq 0$ and $a \neq \pm b$.

Definition

Suppose that $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are agM sequences for a, b. Then:

- b_{n+1} is the right choice if $|a_{n+1} - b_{n+1}| \leq |a_{n+1} + b_{n+1}|$, and if equality occurs, we require $\mathrm{Im}(b_{n+1}/a_{n+1}) > 0$.
- $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are good sequences if b_{n+1} is the right choice for all but finitely many n.

Note that a, b has only countably many good agM sequences.

Theorem

Let $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ be agM sequences of a, b as above. Then:

1. *$\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ always converge to a common limit.*
2. *The common limit is nonzero $\iff$ $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ are good sequences*

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Main Theorem

Definition

- The values of $M(a,b)$ are the common limits of good sequences.
- The simplest value of $M(a,b)$ is where b_{n+1} is the right choice $\forall\, n$.

Theorem

Let $a, b \in \mathbb{C}$ with $a, b \neq 0$, $a \neq \pm b$, and $|a| \geq |b|$. Define

$$\mu = \text{simplest value of } M(a,b), \ \ \lambda = \text{simplest value of } M(a+b, a-b).$$

Then all values μ' of $M(a,b)$ are given by

$$\frac{1}{\mu'} = \frac{d}{\mu} + \frac{ic}{\lambda},$$

where d, c are relatively prime with $d \equiv 1 \bmod 4$, $c \equiv 0 \bmod 4$.

The proof uses theta functions, modular functions, and modular forms!

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Theta Functions

For $\tau \in \mathfrak{h} =$ upper half plane, let $q = e^{\pi i \tau}$ and define the theta functions:

$$p(\tau) = 1 + 2q + 2q^4 + 2q^9 + \cdots$$
$$q(\tau) = 1 - 2q + 2q^4 - 2q^9 + \cdots$$
$$r(\tau) = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots$$

Some Identities

- $p(\tau)^4 - q(\tau)^4 = r(\tau)^4$
- $p(2\tau)^2 = \frac{1}{2}(p(\tau)^2 + q(\tau)^2)$
- $q(2\tau)^2 = p(\tau)q(\tau) = \sqrt{p(\tau)^2 q(\tau)^2}$

Consequences

- $(a, b) = (p(\tau)^2, q(\tau)^2) \Rightarrow (a_1, b_1) = (p(2\tau)^2, q(2\tau)^2)$
- $M(p(\tau)^2, q(\tau)^2) = \lim_{n\to\infty} p(2^n\tau)^2 = \lim_{n\to\infty} q(2^n\tau)^2 = 1$

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For $\tau \in \mathfrak{h} =$ upper half plane, let $q = e^{\pi i \tau}$ and define the theta functions:

$$p(\tau) = 1 + 2q + 2q^4 + 2q^9 + \cdots$$
$$q(\tau) = 1 - 2q + 2q^4 - 2q^9 + \cdots$$
$$r(\tau) = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots$$

Some Identities

- $p(\tau)^4 - q(\tau)^4 = r(\tau)^4$
- $p(2\tau)^2 = \frac{1}{2}(p(\tau)^2 + q(\tau)^2)$
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Consequences

- $(a, b) = (p(\tau)^2, q(\tau)^2) \Rightarrow (a_1, b_1) = (p(2\tau)^2, q(2\tau)^2)$
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Apply to agM

The previous slide implies that $M(\underbrace{\mu p(\tau)^2}_{a}, \underbrace{\mu q(\tau)^2}_{b}) = \mu$.

Now define

$$k'(\tau) = \frac{q(\tau)^2}{p(\tau)^2} \quad (k' \text{ is from the theory of elliptic integrals}).$$

How to Compute Values of $M(a, b)$

- Pick $\tau \in \mathfrak{h}$ such that $\boxed{k'(\tau) = \frac{b}{a}}$ and set $\boxed{\mu = \frac{a}{p(\tau)^2}}$.
- Then $\mu q(\tau)^2 = \frac{a}{p(\tau)^2} q(\tau)^2 = a k'(\tau) = b$.
- It follows that μ is a value of $M(a, b)$.

Theorem

When $a, b \neq 0$, $a \neq \pm b$, $|a| \geq |b|$, all values of $M(a, b)$ arise in this way.

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A Modular Function

Theorem

$k'(\tau)$ is a modular function for the congruence subgroup

$$\Gamma_2(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod 4, b \equiv 0 \bmod 2 \right\}.$$

and induces $k' : \mathfrak{h}/\Gamma_2(4) \simeq \mathbb{C} \setminus \{0, \pm 1\}$ ($\leftarrow \frac{b}{a}$ lives in here).

Consequences

Pick $\tau_0 \in F$ with $k'(\tau_0) = \frac{b}{a}$. Then:

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$$\tau = \frac{a\tau_0 + b}{c\tau_0 + d} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(4).$$

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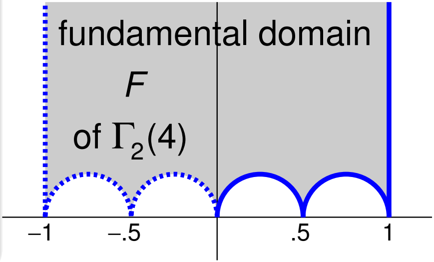

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Proof of the Main Theorem

The values are $\mu' = \frac{a}{p(\tau)^2}$ with $k'(\tau) = \frac{b}{a}$. Then:

$$\frac{1}{\mu'} = \frac{1}{a}p(\tau)^2 = \frac{1}{a}p\left(\frac{a\tau_0 + b}{c\tau_0 + d}\right)^2 = \frac{1}{a}(c\tau_0 + d)p(\tau_0)^2 \text{ since } \Gamma_2(4) \subseteq \Gamma_0(2)$$

$$= d\,\frac{p(\tau_0)^2}{a} + c\,\frac{\tau_0 p(\tau_0)^2}{a}, \quad \mu = \frac{a}{p(\tau_0)^2} = \text{simplest value of } M(a, b)$$

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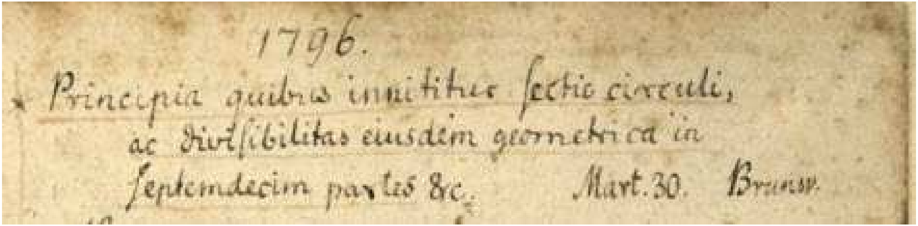

Facts about the diary

- It has 146 entries, dated from March 30, 1796 to July 9,1814.
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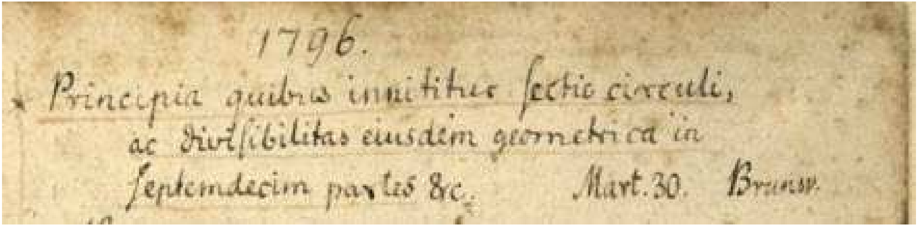

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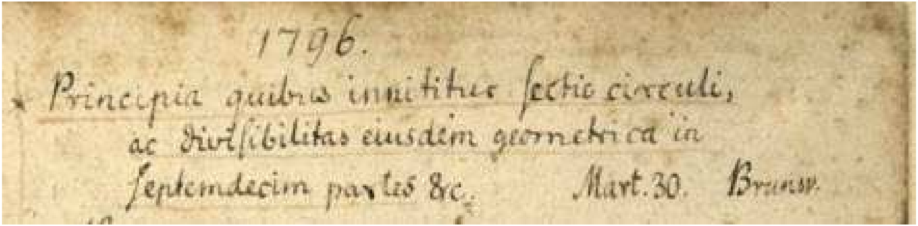

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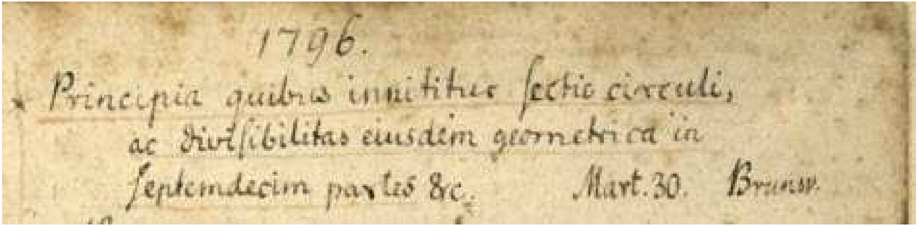

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Entry 98

May 30, 1799

We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is $\frac{\pi}{\varpi}$ to the eleventh decimal place; the demonstration of this fact will surely open an entirely new field of analysis.

From a 1786 paper of Euler: $\begin{cases} \int_0^1 \frac{dz}{\sqrt{1-z^4}} \times \int_0^1 \frac{z^2 dz}{\sqrt{1-z^4}} = \frac{\pi}{4} \\ \int_0^1 \frac{z^2 dz}{\sqrt{1-z^4}} = 0.55907011736 \end{cases}$

Since $\frac{\varpi}{2} = \int_0^1 \frac{dz}{\sqrt{1-z^4}}$, Euler's results imply that

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For the agM, this refers to the countably many values when $a, b \in \mathbb{C}$.

Gauss's Version of the "mutual connection"

The agM changes, when one chooses the negative value for one of n', n'', n''' etc.: however all resulting values are of the following form:

$$\frac{1}{(\mu)} = \frac{1}{\mu} + \frac{4ik}{\lambda}.$$

In this quote, "negative value for one of n', n'', n'''" refers to making bad choices of b_{n+1}. Also, μ is the "einfachste Mittel" (simplest mean).

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A "Most Important" Theorem

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$$M(p(\tau)^2, q(\tau)^2) = 1,$$

i.e, the agM of $p(\tau)^2$ and $q(\tau)^2$ is always equal to 1.

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THEOREM (23):

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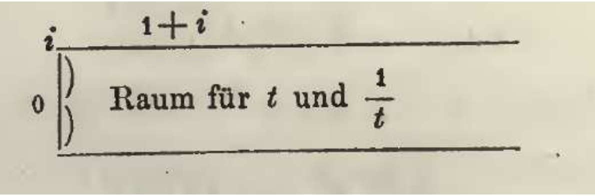

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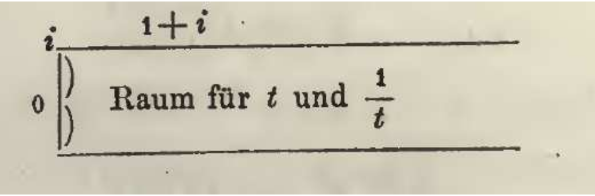

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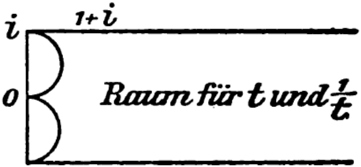

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$$M(\sqrt{2}, 1) = \frac{\pi}{\varpi} \tag{1}$$

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- By December 1799, Gauss had proved (1), which establishes a link between elliptic integrals and the agM.
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- However, when you bring the complex agM into the picture, along with the connections to modular functions and modular forms, then indeed we have a "new field of analysis".
- Gauss kept his promise!

An Entirely New Field of Analysis

- Entry 98 of Gauss's mathematical diary, dated May 30, 1799, claimed that proving

$$M(\sqrt{2}, 1) = \frac{\pi}{\varpi} \tag{1}$$

  would "open an entirely new field of analysis".
- By December 1799, Gauss had proved (1), which establishes a link between elliptic integrals and the agM.
- This is nice but does not constitute a "new field of analysis".
- However, when you bring the complex agM into the picture, along with the connections to modular functions and modular forms, then indeed we have a "new field of analysis".
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Letter to Bessel

March 30, 1828

I shall most likely not soon prepare my investigations on transcendental functions which I have had for many years – since 1798 – because I have many other matters which must be cleared up. Herr Abel has now, I see, anticipated me and relieved me of the burden in regard to one-third of these matters, particularly since he carried out all developments with great concision and elegance.

By "one-third of these matters", Gauss meant his unpublished work on elliptic functions, which I did not discuss in this lecture.

The other two-thirds consist of:

- His work on modular functions and modular forms, and
- His work on the arithmetic-geometric mean!

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