Why is the Riemann Hypothesis Important?

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University of Connecticut

August 11, 2016
Bericht
über die
zur Bekanntmachung geeigneten Verhandlungen
der Königl. Preuß. Akademie der Wissenschaften
zu Berlin
im Monat November 1859.

Vorsitzender Sekretär: Hr. Encke.

Hr. Steiner las über einige allgemeine Bestimmungsarten der Curven und Flächen zweiter Ordnung und daraus folgenden Sätzen.

For $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, set $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$. 
The Zeta-function

A representation of $\zeta(s)$, for $\text{Re}(s) > 1$, as a product over primes:

$$
\prod_p \frac{1}{1 - 1/p^s} = \frac{1}{1 - 1/2^s} \frac{1}{1 - 1/3^s} \frac{1}{1 - 1/5^s} \frac{1}{1 - 1/7^s} \cdots
$$

$$
= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \cdots\right) \left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \cdots\right) \cdots
$$

$$
= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots
$$

$$
= \zeta(s).
$$

This is due to Euler for real $s > 1$ and is called the Euler product.

Euler (1738) first became famous by computing $\zeta(2) = \frac{\pi^2}{6}$.

Letting $s \to 1^+$ makes $\zeta(s) \to \infty$, so by Euler product there are infinitely many primes: the first proof in analytic number theory. Riemann noted that $\zeta(s)$ for $\text{Re}(s) > 1$ is \textbf{analytic} and he analytically continued $\zeta(s)$ to $\mathbb{C} - \{1\}$ using the Gamma function.
The Gamma-function is defined on $\mathbb{C}$ by

$$
\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x} \text{ for } \text{Re}(s) > 0
$$

and $\Gamma(s + 1) = s\Gamma(s)$ for other $s$ (integration by parts). It’s analytic away from $0, -1, -2, \ldots$ and $\Gamma(n + 1) = n!$. 
Analytic continuation of the zeta-function

**Theorem (Riemann)**

The function \( \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \) is analytic on \( \text{Re}(s) > 1 \) and extends to an analytic function on \( \mathbb{C} - \{ 1 \} \). The “completed zeta-function”

\[
Z(s) := \pi^{-s/2} \Gamma\left( \frac{s}{2} \right) \zeta(s)
\]

is analytic on \( \mathbb{C} - \{ 0, 1 \} \) and satisfies the functional equation

\[
Z(1 - s) = Z(s).
\]

**Remark.** The zeta-function itself satisfies the messier functional equation

\[
\zeta(1 - s) = 2(2\pi)^{-s} \Gamma(s) \cos \left( \frac{\pi s}{2} \right) \zeta(s).
\]
Functional equation:

\[ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \]

Not the functional equation:
For $\text{Re}(s) > 1$, $\zeta(s) = \prod_p \frac{1}{1 - 1/p^s} \neq 0$. Also $\pi^{-s/2} \Gamma(s/2) \neq 0$, so $Z(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) \neq 0$ for $\text{Re}(s) > 1$.

Therefore when $\text{Re}(s) < 0$, $Z(s) = Z(1 - s) \neq 0$. Because $\Gamma(s/2)$ blows up at $s = 0, -2, -4, -6, \ldots$ and $Z(s)$ is finite and nonzero at $s = -2, -4, -6, \ldots$, $\zeta(s) = 0$ when $s = -2, -4, -6, \ldots$. These are called the trivial zeros of $\zeta(s)$. 
The zeros of $\zeta(s)$ that are not explained by $\Gamma(s/2)$ are called nontrivial. They are in the *critical strip* $0 \leq \text{Re}(s) \leq 1$.

The center line $\text{Re}(s) = \frac{1}{2}$ is called the *critical line*. Since $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is nonzero outside the critical strip, nontrivial zeros of $\zeta(s) = \text{all zeros of } Z(s)$. First three nontrivial zeros with $\text{Im}(s) \geq 0$ are approximately

\[ \frac{1}{2} + 14.134725i, \quad \frac{1}{2} + 21.022039i, \quad \frac{1}{2} + 25.010857i. \]
Statement of the Riemann Hypothesis

Wurzeln von \( \xi(t) = 0 \), multiplich mit \( 2\pi i \). Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.
Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all non-trivial zeros of the Riemann zeta function \( \zeta(s) \) have real part \( \frac{1}{2} \). Equivalently, all zeros of \( Z(s) = \zeta(\frac{1}{2} + it) \) are real.

What was Riemann's investigation?
Statement of the Riemann Hypothesis


に等しい。この積分は又この領域にある $\xi(t) = 0$ 根ので個数に $2\pi i$ をかけたものに等しい。実際この領域内にはほぼこの程度の多さの実根がある。全ての根が実数であることはかなり確かなることである。もちろんこれについては確かな証明が望まれる。しかし私は、粗雑ながらも若干試みたがよい成果を得なくて、これについての探求を当分の間しなかった。それは私の研究の一番直接の目的にとって無用におもわれたからだ。
the number of roots of \( \xi(t) = 0 \) in the domain multiplied by \( 2\pi i \). One finds in fact about this many real roots within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation.

Here \( \xi(t) \) is essentially \( Z(1/2 + it) \), which is \textbf{real-valued} for real \( t \).

**Riemann Hypothesis:** Nontrivial zeros of \( \zeta(s) \) have \( \text{Re}(s) = \frac{1}{2} \). Equivalently, all zeros of \( Z(s) \) have \( \text{Re}(s) = \frac{1}{2} \): all zeros of \( Z(1/2 + it) \) are real.

What was Riemann’s investigation?
Riemann’s Investigation

Riemann set out a program for proving the prime number theorem:

\[ \pi(x) = \#\{\text{primes} \leq x\} \sim \frac{x}{\log x} \quad \text{as } x \to \infty. \]

This had first been conjectured publicly by Legendre (1796).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\pi(x))</th>
<th>(x / \log x)</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^2)</td>
<td>25</td>
<td>21.7</td>
<td>1.15</td>
</tr>
<tr>
<td>(10^4)</td>
<td>1229</td>
<td>1085.7</td>
<td>1.13</td>
</tr>
<tr>
<td>(10^6)</td>
<td>78498</td>
<td>72382.4</td>
<td>1.08</td>
</tr>
<tr>
<td>(10^8)</td>
<td>5761455</td>
<td>5428681.0</td>
<td>1.06</td>
</tr>
<tr>
<td>(10^{10})</td>
<td>455052511</td>
<td>434294481.9</td>
<td>1.04</td>
</tr>
</tbody>
</table>

Gauss (before Legendre) developed a probabilistic model which suggested to him that

\[ \pi(x) \sim \text{Li}(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}. \]
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<table>
<thead>
<tr>
<th>( x )</th>
<th>( \pi(x) )</th>
<th>( \text{Li}(x) )</th>
<th>( \pi(x)/\text{Li}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^2 )</td>
<td>25</td>
<td>29.0</td>
<td>.859</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>1229</td>
<td>1245.0</td>
<td>.987</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>78498</td>
<td>78626.5</td>
<td>.998</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>5761455</td>
<td>5762208.3</td>
<td>.9998</td>
</tr>
<tr>
<td>( 10^{10} )</td>
<td>455052511</td>
<td>455055613.5</td>
<td>.999993</td>
</tr>
</tbody>
</table>

Gauss (before Legendre) developed a probabilistic model which suggested to him that

\[ \pi(x) \sim \text{Li}(x) := \int_{2}^{x} \frac{dt}{\log t} \sim \frac{x}{\log x}. \]
Riemann compared two product representations of $\zeta(s)$: its Euler product over the primes and its factorization over its zeros:

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s} = \frac{1}{2} \left(\frac{2\pi}{e}\right)^s \frac{1}{s-1} \prod_{\zeta(\rho) = 0} \left(1 - \frac{s}{\rho}\right)^{m_\rho} e^{m_\rho s/\rho}.$$

These two products are the basic reason why information about zeros of $\zeta(s)$ can lead to information about prime numbers! Applying an integral transform to both formulas implies for $x > 2$

$$\sum_{p^k \leq x} \log p = x - \sum_{p^k \leq x} m_\rho \frac{x^\rho}{\rho} - \log(2\pi).$$

The prime number theorem can be expressed in terms of growth of the left side. Since $|x^\rho| = x^{\Re(\rho)}$, estimating size of middle term compared to $x$ needs knowledge of $\Re(\rho)$. 

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**Riemann’s Investigation**

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The proof of the prime number theorem (1896) did not use RH.

**Theorem (Wiener, 1932)**

*The prime number theorem is equiv. to* \( \zeta(s) \neq 0 \) *for* \( \Re(s) = 1 \).

This is what a known zero-free region in upper part of critical strip looks like (for large imaginary parts).
The proof of the prime number theorem (1896) did not use RH.

**Theorem (Wiener, 1932)**

*The prime number theorem is equiv. to \( \zeta(s) \neq 0 \) for \( \text{Re}(s) = 1 \).*

No zero-free region of the form \( \{ \text{Re}(s) > 1 - \varepsilon \} \) is known. Find one and become famous! Too bad \( \text{Re}(s) = 1 \) is not compact.
Using Riemann Hypothesis: $\pi(x)$ vs. $\text{Li}(x)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\pi(x)$</th>
<th>$\text{Li}(x)$</th>
<th>$\pi(x) - \text{Li}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^2$</td>
<td>25</td>
<td>29.0</td>
<td>$-4.0$</td>
</tr>
<tr>
<td>$10^4$</td>
<td>1229</td>
<td>1245.0</td>
<td>$-16.0$</td>
</tr>
<tr>
<td>$10^6$</td>
<td>78498</td>
<td>78626.5</td>
<td>$-128.5$</td>
</tr>
<tr>
<td>$10^8$</td>
<td>5761455</td>
<td>5762208.3</td>
<td>$-753.3$</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>455052511</td>
<td>455055613.5</td>
<td>$-3102.5$</td>
</tr>
</tbody>
</table>

Is $\pi(x) < \text{Li}(x)$ for all $x > 2$? No: Littlewood (1914) showed $\pi(x) > \text{Li}(x)$ infinitely often. What is an example?

Skewes (1933): RH implies $\pi(x) > \text{Li}(x)$ for some $x < 10^{10^{10^{34}}}$. 

Skewes (1955): $\pi(x) > \text{Li}(x)$ for some $x < 10^{10^{10^{963}}}$ (RH not used). This upper bound has been reduced to around $10^{316}$, must be above $10^{19}$. Still no example is known.
**Conjecture:** The nontrivial zeros of $\zeta(s)$ are simple.

Here is a question of Polya which involves this conjecture.

Set $\text{Even}(x) = \# \{ n \leq x : n \text{ has an even number of prime factors} \}$ and $\text{Odd}(x) = \# \{ n \leq x : n \text{ has an odd number of prime factors} \}$. For example, $12 = 2^2 \cdot 3$ has three prime factors.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even($x$)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Odd($x$)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>5</td>
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</tbody>
</table>

- Polya (1919) conjectured $\text{Even}(x) \leq \text{Odd}(x)$ for all $x \geq 2$.
- Ingham (1942) showed this implies RH and simplicity of nontrivial zeros of $\zeta(s)$ and **Q-linear dependence** of positive imaginary parts of the zeros. Unlikely!
- Least counterexample is $x = 906, 150, 257$ (Tanaka, 1980).
Why Believe RH?

Here are two pieces of evidence in the direction of RH:

- Numerical evidence looks impressive, e.g., Gourdon–Demichel (2004) checked first $10^{13}$ zeros of $\zeta(s)$ are on critical line.
- For $\varepsilon > 0$, 0% of nontrivial zeros of $\zeta(s)$ satisfy $\text{Re}(s) \geq \frac{1}{2} + \varepsilon$.

Hilbert’s 23 problems and the Clay Millenium Prize problems both include RH, and the official problem descriptions both emphasize that RH is the simplest case of a generalized Riemann hypothesis for functions resembling $\zeta(s)$. The generalized Riemann hypothesis is much more important for applications than the Riemann hypothesis!

What are other zeta-like functions and concrete problems that they can be applied to?
Generalizing the zeta-function

There are many functions similar to $\zeta(s)$ which each have, or are expected to have, the following properties:

1. a series expansion $f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ for $\text{Re}(s) > 1$,

2. an Euler product of some degree $d \geq 1$: for $\text{Re}(s) > 1$,

\[ f(s) = \prod_p \frac{1}{1 + c_{p,1}/p^s + \cdots + c_{p,d}/p^{ds}}, \]

3. analytic continuation to $\mathbb{C}$ or $\mathbb{C} - \{1\}$ (with suitable growth),

4. a completed function

\[ F(s) = A^s \prod_{i=1}^m \Gamma(\lambda_i s + \mu_i) f(s) \]

satisfying $F(1 - s) = \overline{wF(\overline{s})}$ where $|w| = 1$,

5. **Generalized Riemann Hypothesis (GRH)**: all nontrivial zeros of $f(s)$ are on the line $\text{Re}(s) = \frac{1}{2}$. 
A Dirichlet $L$-function is any infinite series of the form $\sum_{n \geq 1} \frac{\chi(n)}{n^s}$, where $\{\chi(n)\}$ is a periodic and totally multiplicative sequence with $\chi(1) = 1$. When $\chi(n) = 1$ for all $n$ the series is $\zeta(s)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_4(n)$</td>
<td>1</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5(n)$</td>
<td>1</td>
<td>$i$</td>
<td>$-i$</td>
<td>−1</td>
<td>0</td>
<td>1</td>
<td>$i$</td>
<td>$-i$</td>
<td>−1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi^2_5(n)$</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

We set

$$L(s, \chi) := \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)/p^s}.$$  

For $\text{Re}(s) > 1$, the series and product converge and $L(s, \chi) \neq 0$. 
Functional Equation for $L(s, \chi_5)$

For $\text{Re}(s) > 1$,

$$L(s, \chi_5) = 1 + \frac{i}{2^s} - \frac{i}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \frac{i}{7^s} + \cdots.$$ 

The function $\Lambda(s, \chi_5) = \frac{5^{s/2}}{2\pi} \frac{\Gamma(s + 1/2)}{\Gamma(s)} L(s, \chi_5)$ is analytic on $\mathbb{C}$ (no problem at $s = 0, 1$) and satisfies the functional equation

$$\Lambda(1 - s, \chi_5) = w \overline{\Lambda(s, \chi_5)},$$

where $w = \sqrt[4]{-(3 + 4i)/5} \approx .85 - .52i$; $|w| = 1$.

The function $L(s, \chi_5)$ is nonzero for $\text{Re}(s) > 1$ by its Euler product. From the functional equation for $\Lambda(s, \chi_5)$, $L(s, \chi_5) = 0$ at $s = -1, -3, -5, \ldots$, which are called the trivial zeros of $L(s, \chi_5)$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_5(n)$</td>
<td>1</td>
<td>$i$</td>
<td>$-i$</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>$i$</td>
<td>$-i$</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>
Generalized Riemann Hypothesis for Dirichlet $L$-functions

Any $L(s, \chi)$ has a completed form $\Lambda(s, \chi)$ satisfying a functional equation $\Lambda(1 - s, \chi) = w\overline{\Lambda(s, \chi)}$, where $|w| = 1$.

**Generalized Riemann Hypothesis** for $L(s, \chi)$: all nontrivial zeros of $L(s, \chi)$ (equiv., all zeros of $\Lambda(s, \chi)$) are on the line $\text{Re}(s) = \frac{1}{2}$.

<table>
<thead>
<tr>
<th>1</th>
<th>$\chi_4$</th>
<th>$\chi_5$</th>
<th>$\chi_5^2$</th>
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</thead>
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<tr>
<td>$\frac{1}{2} + 14.1347i$</td>
<td>$\frac{1}{2} + 6.0209i$</td>
<td>$\frac{1}{2} + 6.1835i$</td>
<td>$\frac{1}{2} + 6.6484i$</td>
</tr>
<tr>
<td>$\frac{1}{2} + 21.0220i$</td>
<td>$\frac{1}{2} + 10.2437i$</td>
<td>$\frac{1}{2} + 8.4572i$</td>
<td>$\frac{1}{2} + 9.8314i$</td>
</tr>
<tr>
<td>$\frac{1}{2} + 25.0108i$</td>
<td>$\frac{1}{2} + 12.5880i$</td>
<td>$\frac{1}{2} + 12.6749i$</td>
<td>$\frac{1}{2} + 11.9588i$</td>
</tr>
</tbody>
</table>
Even Goldbach conjecture: all even \( n \geq 4 \) are a sum of 2 primes.
Odd Goldbach conjecture: all odd \( n \geq 7 \) are a sum of 3 primes.

Even conj. implies odd conj. \((n\text{ odd implies } n - 3 \text{ is even})\).

**Theorem (Hardy–Littlewood, 1923)**

*Odd Goldbach conjecture true for \( n \gg 0 \) if all Dirichlet L-functions are nonzero on a common right half-plane \( \text{Re}(s) \geq \frac{3}{4} - \varepsilon \).*

**Theorem (Vinogradov, 1937)**

*Odd Goldbach conjecture true for \( n \gg 0 \) unconditionally.*

**Theorem (Deshouillers–Effinger–te Riele–Zinoviev, 1997)**

*Odd Goldbach conjecture true for \( n \geq 7 \) under GRH for all Dirichlet L-functions.*

In the last theorem, GRH used for \( n > 10^{20} \), computers for rest. Helfgott (2013) settled odd Goldbach without GRH.
**Theorem (Miller, 1976)**

*There is a polynomial-time primality test if GRH is true for all Dirichlet $L$-functions.*

It would suffice to have **common** zero-free region $\text{Re}(s) > 1 - \varepsilon$ for all Dirichlet $L$-functions. The run-time for Miller’s test can be made explicit in terms of $\varepsilon$.

**Theorem (Agrawal–Kayal–Saxena, 2002)**

*There is a polynomial-time primality test, unconditionally.*

The AKS test is not Miller’s test. Still not known that Miller’s test runs in polynomial time without assuming GRH.
The functions which should satisfy GRH include

- Riemann zeta-function \( \zeta(s) \),
- Dirichlet \( L \)-function \( L(s, \chi) \),
- Zeta-function of a number field (like \( \mathbb{Q}(i) \) or \( \mathbb{Q}(\sqrt[3]{2}) \)),
- \( L \)-function of a modular form,
- \( L \)-function of elliptic curve over \( \mathbb{Q} \),
- \( L \)-function of representation of a Galois group \( \text{Gal}(K/\mathbb{Q}) \).

It is not just GRH that is an open problem here: analytic continuation to \( \mathbb{C} \) and a functional equation can be a hard theorem or still be unknown!

For example, the analytic continuation and functional equation of \( L \)-functions of elliptic curves over \( \mathbb{Q} \), in general, were proved by the ideas going into Wiles’s proof of Fermat’s Last Theorem.
The \( L \)-function of a modular form

If \( f = \sum_{n \geq 0} a_n q^n \) is a modular form of weight \( k \), its \( L \)-function is

\[
L(s, f) = \sum_{n \geq 1} \frac{a_n}{n^s}.
\]

This does not involve the constant term of \( f \). Since \( a_n = O(n^{k-1}) \), \( L(s, f) \) converges absolutely for \( \text{Re}(s) > k \).

**Theorem**

The completed \( L \)-function \( \Lambda(s, f) = (2\pi)^{-s} \Gamma(s)L(s, f) \) has a meromorphic continuation to \( \mathbb{C} \) except for simple poles at \( s = 0 \) and \( s = k \) with residues \( \pm a_0 \), and \( \Lambda(k - s, f) = (-1)^{k/2} \Lambda(s, f) \).

We expect \( \Lambda(s, f) \) to satisfy the Riemann hypothesis (all its zeros are on \( \text{Re}(s) = k/2 \)) if \( f \) is normalized cuspidal eigenform for the Hecke operators, which is when \( L(s, f) \) has Euler product.

**Example.** Eigenform \( \Delta(\tau) \) has weight 12 and first three zeros of \( \Lambda(s, \Delta(\tau)) \) are \( 6 + it \) for \( t \approx 9.223, 13.907, 17.442 \).
Using GRH: Artin’s Primitive Root Conjecture

**Theorem.** (Gauss) For each prime $p$, the group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.

**Example.** The group $(\mathbb{Z}/7\mathbb{Z})^\times = \{1, 2, 3, 4, 5, 6\}$ has generator $3$: 

$$
3^1 = 3, \quad 3^2 = 2, \quad 3^3 = 6, \quad 3^4 = 4, \quad 3^5 = 5, \quad 3^6 = 1.
$$

**Question.** If $a \in \mathbb{Z}$ is not 0 or $\pm 1$, is $a \mod p$ a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$ for infinitely many primes $p$?

**Example.** Gauss conjectured for $a = 10$ the answer is “Yes”.

Elementary school meaning: $\frac{1}{p}$ has decimal period $p - 1$ infinitely often (try $p = 7$).

**Nonexample.** If $a = 9$ then $a^{(p-1)/2} = 3^{p-1} \equiv 1 \mod p$ when $p \neq 3$, so answer is “No” when $a = 9$.

**Conjecture.** (Artin, 1927) If $a \in \mathbb{Z}$ is not 0, $-1$, or a perfect square then $a \mod p$ generates $(\mathbb{Z}/p\mathbb{Z})^\times$ for infinitely many $p$. 
Theorem (Hooley, 1967)

For each $a \neq 0, -1, \text{ or a perfect square}$, Artin’s conjecture for $a$ is correct if GRH is true for a specific family of zeta-functions of number fields depending on $a$.

The GRH-dependence was later relaxed by Gupta, Murty, Murty, Heath-Brown.

**Example.** Unconditionally, at least one of $a = 2, 3, \text{ or 5}$ is a generator for $(\mathbb{Z}/p\mathbb{Z})^\times$ for infinitely many $p$. (Surely for all three!)

Although Artin’s conjecture (as stated here) has been proved true for most $a$ unconditionally it is not known for any one specific $a$. 
An elliptic curve over $\mathbb{Q}$ is a smooth curve with equation $E : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$. Rational solutions $E(\mathbb{Q})$ are an abelian group.

**Example.** On $y^2 = x^3 - 2x$, set $P = (2, 2)$ and $Q = (0, 0)$. 

![Diagram of an elliptic curve with points P and Q and their sum P + Q.]
An elliptic curve over $\mathbb{Q}$ is a smooth curve with equation $E : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$. Rational solutions $E(\mathbb{Q})$ are an abelian group.

**Example.** On $y^2 = x^3 - 2x$, set $P = (2, 2)$ and $Q = (0, 0)$.
Using GRH: Elliptic Artin Conjecture

For most primes $p$, the reduced equation

$$E : y^2 \equiv x^3 + ax + b \mod p$$

is an elliptic curve over $\mathbb{Z}/p\mathbb{Z}$ and its solutions $E(\mathbb{Z}/p\mathbb{Z})$ form a finite abelian group.

The collection of groups $E(\mathbb{Z}/p\mathbb{Z})$ is analogous to the collection of groups $(\mathbb{Z}/p\mathbb{Z})^\times$.

**Question** (Lang–Trotter): If a point $P \in E(\mathbb{Q})$ has infinite order, does it generate $E(\mathbb{Z}/p\mathbb{Z})$ for infinitely many $p$?

**Example**: Let $E : y^2 = x^3 - 2x$ and $P = (2, 2)$. Working modulo $3$, $E(\mathbb{Z}/3\mathbb{Z}) = \{(0, 0), (2, 1), (2, 2) \mod 3\}$. This is $2P, 3P, P$. The group $E(\mathbb{Z}/p\mathbb{Z})$ is generated by $P$ for $p = 3, 5, 11, 19, \ldots$, although it’s not known that the list continues indefinitely.
Are we even certain that the groups $E(\mathbb{Z}/p\mathbb{Z})$ are cyclic for infinitely many $p$? (Compare to: $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic for all $p$.)

**Obstruction:** If $x^3 + ax + b$ has all rational roots then for $p \gg 0$ the group of solutions to $y^2 \equiv x^3 + ax + b \mod p$ includes more than one point of order 2, so it is not cyclic.

**Example.** For the elliptic curve $E : y^2 = x^3 - x$, $E(\mathbb{Z}/p\mathbb{Z})$ is not cyclic for any $p > 2$.

**Theorem (Serre)**

*If $E(\mathbb{Q})$ is infinite, $x^3 + ax + b$ has irrational root, and GRH is true for zeta-functions of a specific family of number fields depending on $E$, then $E(\mathbb{Z}/p\mathbb{Z})$ is cyclic for infinitely many $p$.*

Gupta and Murty later proved this unconditionally (no use of GRH).
**Conjecture** (Lang–Trotter). If $P \in E(\mathbb{Q})$ has infinite order and $x^3 + ax + b$ has an irrational root then $P$ generates $E(\mathbb{Z}/p\mathbb{Z})$ for infinitely many $p$.

**Theorem (Serre)**

The Lang–Trotter conjecture for $E$ and $P$ is true if GRH is true for the zeta-functions of a specific family of number fields depending on $E$ and $P$.

Without using GRH, Gupta and Murty showed for certain $E$ that some point in $E(\mathbb{Q})$ is a generator of $E(\mathbb{Z}/p\mathbb{Z})$ for infinitely many $p$. The ideas here led to their work on the original Artin conjecture (e.g., 2, 3, or 5 generates $(\mathbb{Z}/p\mathbb{Z})^\times$ infinitely often).
Why Believe GRH (Besides Numerical Data)?

- It has a proved analogue in characteristic $p$.

**Example.** Zeta-function of curve $y^2 = x^3 - 2x$ over $\mathbb{Z}/5\mathbb{Z}$ is

$$
\frac{1 - 4 \cdot 5^{-s} + 5 \cdot 5^{-2s}}{(1 - 5^{-s})(1 - 5 \cdot 5^{-s})} = \frac{(1 - (2 + i)5^{-s})(1 - (2 - i)5^{-s})}{(1 - 5^{-s})(1 - 5 \cdot 5^{-s})},
$$

so $s$ is zero $\Rightarrow 5^s = 2 \pm i \Rightarrow 5^{\text{Re}(s)} = \sqrt{5} \Rightarrow \text{Re}(s) = \frac{1}{2}$.

- The proof of GRH in characteristic $p$ uses an interpretation of the zeros as eigenvalues of a special operator (and substantial ideas from algebraic geometry).

- Hilbert and Polya asked if there is a self-adjoint operator on a Hilbert space whose eigenvalues are the zeros of $Z(1/2 + it)$. Eigenvalues of a self-adjoint operator are real.

- Meyer (2006), following work of Connes (1999), introduced spaces (though not Hilbert spaces) and operators whose eigenvalues correspond naturally to the zeros of functions for which GRH is expected.
**Warning:** GRH can’t be a formal consequence of an analytic continuation and functional equation “of zeta-type”.

Let $Q(x, y) = x^2 + 5y^2$, so $Q(x, y) > 0$ for $(x, y) \neq (0, 0)$. The zeta-function of $Q$ is defined to be

$$
\zeta_Q(s) = \sum_{(x,y) \in \mathbb{Z}^2} \frac{1}{Q(x,y)^s} = \sum_{(x,y) \in \mathbb{Z}^2} \frac{1}{(x^2 + 5y^2)^s},
$$

for $\text{Re}(s) > 1$. It is analytic there.

The product $Z_Q(s) = 20^{s/2}(2\pi)^{-s}\Gamma(s)\zeta_Q(s)$ extends analytically to $\mathbb{C} - \{0, 1\}$ and satisfies the functional equation

$$
Z_Q(1 - s) = Z_Q(s).
$$
H. S. A. Potter and E. C. Titchmarsh (1935): the function $\zeta_Q(s)$ has zeros on $\text{Re}(s) = 1/2$ and also zeros with $\text{Re}(s) > \frac{1}{2}$ or even $\text{Re}(s) > 1$. (No Euler product!)
H. S. A. Potter and E. C. Titchmarsh (1935): the function $\zeta_Q(s)$ has zeros on $\text{Re}(s) = 1/2$ and also zeros with $\text{Re}(s) > \frac{1}{2}$ or even $\text{Re}(s) > 1$. (No Euler product!)

- Ph.D. at Oxford under Titchmarsh on $\zeta_Q(s)$, 1933?
- Member IAS, 1933–1935.
- Univ. Aberdeen, 1936–retirement.
- President of Edinburgh Mathematical Society, 1953.
The prime numbers are related to the zeros of $\zeta(s)$ because $\zeta(s)$ can be written as a product in two ways: over the primes and over its zeros.

RH has its own applications, but it is important more because it is the simplest case of GRH.

Most applications of GRH require it for *infinite* families.

A zero-free region of the form $\Re(s) > 1 - \varepsilon$ would be a huge advance and have applications.

GRH has been an excellent guide to what is probably true: theorems proved with it may later be proved unconditionally by other methods.
Questions?
Since \( \zeta(s) \), \( \pi^{-s/2} \), and \( \Gamma(s/2) \) are real-valued for real \( s > 1 \), they all satisfy \( f(s) = \overline{f(s)} \), so \( Z(s) = \overline{Z(s)} \). If \( s = 1/2 + it \) for real \( t \),

\[
Z \left( \frac{1}{2} + it \right) = Z \left( \frac{1}{2} + it \right)
\]

\[
= Z \left( \frac{1}{2} - it \right)
\]

\[
= Z \left( 1 - \left( \frac{1}{2} - it \right) \right)
\]

\[
= Z \left( \frac{1}{2} + it \right).
\]

Therefore \( Z(s) \) is \textit{real-valued} on the critical line. Nontrivial zeros of \( \zeta(s) \) on the critical line can be detected by finding sign changes in \( Z(1/2 + it) \). May assume \( t \geq 0 \) since \( Z(1/2 - it) = Z(1/2 + it) \).
To count nontrivial zeros in critical strip up to height $T$

1. integrate $(s(1 - s)Z(s))'/(s(1 - s)Z(s))$ around region to count zeros of $Z(s)$ inside,

2. compute $Z(1/2 + it)$ for $0 < t < T$ and count sign changes,

3. when the two numbers match, we have proved the Riemann hypothesis up to height $T$.

The contour integral counts multiple zeros multiply often, so this method only works if all nontrivial zeros are simple.
For a completed Dirichlet $L$-function $\Lambda(s, \chi)$, whose zeros are the nontrivial zeros of $L(s, \chi)$, the functional equation is

$$\Lambda(1 - s, \chi) = w \overline{\Lambda(s, \chi)},$$

with $|w| = 1$.

Writing $w = u^2$, the functional equation implies $\frac{1}{u} \Lambda(s, \chi)$ is real-valued on the critical line, so its simple zeros on that line can be found by locating sign changes. Therefore a process like that for $Z(s)$ can be used.