PROJECT ON COMPUTING THE U_p-EIGENVALUES OF FAMILIES OF OVERCONVERGENT AUTOMORPHIC FORMS

Abstract. In this project, we use SAGE to compute the U_p-eigenvalues of families of overconvergent automorphic forms on a definite quaternion algebra.

The style of writing is: we write general theory in black and our example in blue.

Acknowledgments. The entire project is originated from the thesis of Daniel Jacobs [Ja04], which was later further refined by Daqing Wan, L.X., and Jun Zhang in [WXZ14$^+$]. We thank them for the foundational works.

1. Automorphic forms on a definite quaternion algebra

1.1. **The quaternion algebra.** Let D be a definite quaternion algebra over $\mathbb{Q}$ which splits at a fixed prime p. As our first example which we shall test, we take

$$D = \mathbb{Q}\langle \mathbf{i}, \mathbf{j}\rangle/(\mathbf{i}^2 = \mathbf{j}^2 = -1, \mathbf{ij} = -\mathbf{ji}),$$

and $p = 3$. We often put $\mathbf{k} = \mathbf{ij}$ so that $\mathbf{i} = \mathbf{jk} = -\mathbf{kj}$ and $\mathbf{j} = \mathbf{ki} = -\mathbf{ik}$.

We need to introduce some local information of D.

- $D \otimes \mathbb{R}$ is isomorphic to the Hamiltonian quaternion $\mathbf{H}$. For our example, $D \otimes \mathbb{R} \cong \mathbb{R}\langle \mathbf{i}, \mathbf{j}\rangle \cong \mathbf{H}$.
- There is a finite set Σ_D of prime numbers such that a prime ℓ does *not* belong to Σ_D if and only if $D \otimes \mathbb{Q}_\ell \cong \mathrm{M}_2(\mathbb{Q}_\ell)$ (and we fix such an isomorphism). This set Σ_D always consists of odd number of primes; these are called ramified primes. Our assumption on D requires that $p \notin \Sigma_D$. For our example, $\Sigma_D = \{2\}$, i.e. our D is only ramified at 2 (and at ∞).

  For later application, we need to specify an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{Q}_3 \cong \mathrm{M}_2(\mathbb{Q}_3)$. We take it so that

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \leftrightarrow \begin{pmatrix} \nu_3 & 1 \\ 1 & -\nu_3 \end{pmatrix}, \quad \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \mathbf{k} \leftrightarrow \begin{pmatrix} 1 & -\nu_3 \\ -\nu_3 & -1 \end{pmatrix}, \tag{1.1}$$

  where ν_3 is the square root of -2 that is congruent to 1 modulo 3; explicitly, we have a 3-adic expansion

$$\nu_3 = 1 + 3 + 2 \cdot 3^2 + 2 \cdot 3^5 + 3^7 + \cdots .$$

- For $\ell \in \Sigma_D$, $D \otimes \mathbb{Q}_\ell$ is a division algebra (or a non-commutative field). Explicitly, we can write it as $\mathbb{Q}_{\ell^2}(\varpi)$, where $\mathbb{Q}_{\ell^2}$ is the unique unramified extension of $\mathbb{Q}_\ell$, and ϖ is an element such that $\varpi^2 = \ell$ and $\varpi a = \sigma(a)\varpi$ for all $a \in \mathbb{Q}_{\ell^2}$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\ell^2}/\mathbb{Q}_\ell)$. For our example, $D \otimes \mathbb{Q}_2 = \mathbb{Q}_2\langle \mathbf{i}, \mathbf{j}\rangle$ is a non-commutative field, such that $(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^{-1} = \frac{a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}}{a^2+b^2+c^2+d^2}$ if $a, b, c, d \in \mathbb{Q}_2$ not all zero. (The upshot is that whenever $a, b, c, d \in \mathbb{Q}_2$ not all zero, $a^2 + b^2 + c^2 + d^2 \neq 0$.)

The key statement we introduce here is the Jacquet–Langlands correspondence, which roughly says that a large part of the information regarding modular forms can be seen on this definite quaternion algebra. It gives an isomorphism

$$S_k(\Gamma_0(Np^m);\psi_m)^{\Sigma_D\text{-new}} \cong S_k^D(K_0(Np^m);\psi_m), \tag{1.2}$$

where the left hand side is the space of Σ_D-new forms of the given level, and the right hand side is the automorphic forms on D, and the isomorphism preserves the action of all Hecke operators. For the particular case appearing in our project, we look at the isomorphism

$$S_k(\Gamma_0(18);\psi_9)^{2\text{-new}} \cong S_k^D(K_0(18);\psi_9). \tag{1.3}$$

We now explain the terms involved in this isomorphism.

1.2. **Level structure for modular forms.** We need to interpret the level structure of modular forms as open compact subgroups of $(D\otimes\mathbb{A}_f)^\times$, where $\mathbb{A}_f$ is the ring of finite adeles of $\mathbb{Q}$. For the purpose of this project, we limit our consideration to level structures of the following kind.

- We choose an integer N such that
  - N is square-free,
  - p does not divide N, and
  - for every prime ℓ in Σ_D, $\ell | N$.

  This is the prime-to-p part of the level for modular forms. In our example, we require N to be an even square-free number, or rather just $N = 2$.
- for the level at p, we take $\Gamma_0(p^m)$-level (with $m \geq 1$) with a nebentypus character $\psi_m : (\mathbb{Z}/p^m\mathbb{Z})^\times \to \mathbb{Q}_p(\zeta_{p^{m-1}})^\times$ of conductor p^m (namely doesn't factor through $(\mathbb{Z}/p^{m-1}\mathbb{Z})^\times$) (in our case, we consider the $\Gamma_0(9)$-level with nebentypus character $(\mathbb{Z}/9\mathbb{Z})^\times \to \mathbb{Q}(\zeta_3)^\times$, sending 2 to ζ_6).

So on the modular forms side, we consider the level group $\Gamma_0(Np^m)$ with Nebentypus character $\psi_m : (\mathbb{Z}/Np^m\mathbb{Z})^\times \to (\mathbb{Z}/p^m\mathbb{Z})^\times \to \mathbb{Q}_p(\zeta_{p^{m-1}})^\times$. Namely, we look at $S_k(\Gamma_0(Np^m);\psi_m)$ consisting of modular cuspforms that satisfies

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^k\psi_m(d)f(z)$$

with $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $Np^m|c$.

Here and after, we implicitly assume the following

Hypothesis 1.3. The weight k satisfies $k \geq 2$ and $\psi_m(-1)\cdot(-1)^k = 1$.

1.4. **Hecke operators.** For each prime $\ell \nmid Np$, there is a well-defined Hecke operator T_ℓ acting on the space of modular forms $S_k(\Gamma_0(Np^m);\psi_m)$. More precisely, for $f(q) = \sum_{n=1}^\infty a_n q^n$, we put

$$T_\ell(f)(q) := \sum_{n=1}^\infty a_{\ell n}q^n + \psi_m(\ell)\ell^{k-1}\sum_{n=1}^\infty a_n q^{\ell n}.$$

Different Hecke operators commutes with each other, and as a fact that they act semisimply on the space of modular forms $S_k(\Gamma_0(Np^m);\psi_m)$ (namely, the generalized eigenspaces are eigenspaces).

There is a distinguished Hecke operator U_p acting on the $S_k(\Gamma_0(Np^m);\psi_m)$ such that for $f(q)=\sum_{n=1}^{\infty} a_n q^n$,

$$U_p(f)(q) := \sum_{n=1}^{\infty} a_{pn} q^n.$$

The action of U_p commutes with all other Hecke operators T_ℓ. (For a very special reason , namely ψ_m has conductor exactly p^m, the action of U_p is also semisimple.)

Using the commuting actions of U_p and T_ℓ for all $\ell \nmid Np$, we may decompose $S_k(\Gamma_0(Np^m);\psi_m)$ into eigenspaces

$$S_k(\Gamma_0(Np^m);\psi_m) \cong \bigoplus_{\pi} S_\pi, \tag{1.4}$$

where π denotes a collection of eigenvalues for the operators U_p and T_ℓ, and S_π the corresponding eigenspaces.

A modular form $f \in S_\pi$ is called an *eigenform*; it is called normalized if $a_1(f)=1$. In this case the eigenvalue for the operator U_p is exactly $a_p(f)$, and the eigenvalue for the operator T_ℓ is exactly $a_\ell(f)$.

1.5. **New form theory.** For each prime number $\ell \in \Sigma_D$ (and hence a divisor of N), there are two natural embeddings

$$i_\ell^{(1)}, i_\ell^{(2)} : S_k(\Gamma_0(Np^m/\ell);\psi_m) \longrightarrow S_k(\Gamma_0(Np^m);\psi_m)$$

$$i_\ell^{(1)}(f)(z) = f(z) \quad \text{and} \quad i_\ell^{(2)}(f)(z) = f(\ell z).$$

The sum of the images $\mathrm{Im}(i_\ell^{(1)})+\mathrm{Im}(i_\ell^{(2)})$ is called the space of ℓ-*old forms* of $S_k(\Gamma_0(Np^m);\psi_m)$, denoted by $(S_k(\Gamma_0(Np^m);\psi_m)^{\ell\text{-old}}$.

It turns out that $(S_k(\Gamma_0(Np^m);\psi_m)^{\ell\text{-old}}$ is the direct sum of some factors S_π appearing in (1.4). The direct sum of other factors is called the space of ℓ-*new forms*.

Applying this construction to all $\ell \mid \Sigma_D$, we define the space of Σ_D-*new forms* to be

$$S_k(\Gamma_0(Np^m);\psi_m)^{\Sigma_D\text{-new}} := \bigcap_{\ell\in\Sigma_D} S_k(\Gamma_0(Np^m);\psi_m)^{\ell\text{-new}};$$

it is the direct sum of those factors S_π in (1.4) that do not appear in the image of $i_\ell^{(1)}$ or $i_\ell^{(2)}$ for all $\ell \in \Sigma_D$ (nor in the sum of all these images).

In the example we consider, there are two embeddings

$$i_2^{(1)}, i_2^{(2)} : S_k(\Gamma_0(9);\psi_9) \to S_k(\Gamma_0(18);\psi_9),$$

which induces a *direct* sum decomposition:

$$S_k(\Gamma_0(18);\psi_9) = \mathrm{Im}(i_2^{(1)}) \oplus \mathrm{Im}(i_2^{(2)}) \oplus S_k(\Gamma_0(18);\psi_9)^{2\text{-new}}$$

respecting the action of U_3 and all other T_ℓ for primes $\ell \neq 2,3$.

1.6. **Coefficients.** A key feature of modular forms that allows us to do arithmetic with it is that the space of modular forms $S_k(\Gamma_0(Np^m);\psi_m)$ admits a basis consisting of modular forms whose q-expansions have coefficients in $\mathbb{Q}(\psi_m)$. So it makes sense to talk about modular forms with coefficients in $\mathbb{Q}(\psi_m)$ and hence modular forms with coefficients in $\mathbb{Q}_p(\psi_m) := \mathbb{Q}_p(\mathrm{Im}(\psi_m)) = \mathbb{Q}_p(\zeta_{p^{m-1}})$.

From now on, when we write $S_k(\Gamma_0(Np^m);\psi_m)$, we think of modular forms with coefficients in $\mathbb{Q}_p(\psi_m)$.

We now complete the explanation of the left hand side of the isomorphism (1.2) (and (1.3)). We now turn to defining the space of automorphic forms on D.

1.7. **Adelic groups.** To properly define automorphic forms on D, we need to take the adelic setup. We begin with describing the adelic group $(D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times$.

The ring of finite adeles $\mathbb{A}_f$ is a subring of the direct product $\prod_{\ell \text{ prime}}\mathbb{Q}_\ell$ given explicitly as

$$\mathbb{A}_f := \Big\{(x_\ell)_\ell \in \prod_{\ell \text{ prime}} \mathbb{Q}_\ell \;\Big|\; \text{all but finitely many } x_\ell \in \mathbb{Z}_\ell\Big\}.$$

It is topological ring in which subgroups of the form $\prod_{\ell \text{ prime}} \ell^{a_\ell}\mathbb{Z}_\ell$, where each $a_\ell \in \mathbb{Z}_{\geq 0}$ and only finitely many a_ℓ are nonzero, form a system of open neighborhood of 1.

In a similar manner, recall that $D\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \cong \mathrm{M}_2(\mathbb{Q}_\ell)$ for all $\ell\notin\Sigma_D$ (and we fix such an isomorphism for each ℓ and identify them).[1] Then

$$(D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times := \Big\{(x_\ell)_\ell \in \prod_{\ell \text{ prime}} (D\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)^\times \;\Big|\; \text{all but finitely many } x_\ell \in \mathrm{GL}_2(\mathbb{Z}_\ell)\Big\}.$$

Note that the condition does not make sense for $\ell\in\Sigma_D$ but that only involves finitely many primes and hence does not affect the effect of the statement.

The adelic group $(D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times$ is a topological group such that the subgroups of the form $\prod_{\ell \text{ prime}} K_\ell$, where $K_\ell \subseteq (D\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)^\times$ open and $K_\ell \cong \mathrm{GL}_2(\mathbb{Z}_\ell)$ for all but finitely many ℓ, form a system of open neighborhood of 1. Such subgroup $\prod_{\ell \text{ prime}} K_\ell$ is compact if (and only if) each K_ℓ is.

1.8. **Level subgroups.** We now interpret the level structure on modular forms in terms of open compact subgroups of $(D\otimes\mathbb{A}_f)^\times$. Recall that we consider only a special type of level structure: $\Gamma_0(Np^m)$, where N is a square-free integer divisible by all primes in Σ_D. Accordingly, we take the open compact subgroup of $(D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times$ to be the following.

- If $\ell\nmid Np$, we take $K_\ell := \mathrm{GL}_2(\mathbb{Z}_\ell)$.
- If $\ell \mid N$ but $\ell\notin\Sigma_D$, we take
$$K_\ell := \begin{pmatrix} \mathbb{Z}_\ell^\times & \mathbb{Z}_\ell \\ \ell\mathbb{Z}_\ell & \mathbb{Z}_\ell^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{Z}_\ell).$$
- If $\ell = p$, we take
$$K_p := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{Z}_p).$$
- If $\ell\in\Sigma_D$, the multiplicative division algebra $D\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \cong \mathbb{Q}_{\ell^2}\langle\varpi\rangle$, we take $K_\ell = \mathbb{Z}_{\ell^2}\langle\varpi\rangle^\times = \mathbb{Z}_{\ell^2}^\times + \varpi\mathbb{Z}_{\ell^2}\langle\varpi\rangle$.

[1] Rigorously speaking, we should first take an order $\mathcal{O}_D$ of D and then identify $\mathcal{O}_D\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$ with $\mathrm{M}_2(\mathbb{Z}_\ell)$ for all but finitely many ℓ. This way, we keep the global integral structure up to finitely many primes.

We denote the total product $\prod_{\ell \in \text{ prime}} K_\ell$ as $K_0(Np^m)$.

In the our particular example, we have

$$K_0(18) = \mathbb{Z}_4\langle\varpi_2\rangle^\times \times \begin{pmatrix} \mathbb{Z}_3^\times & \mathbb{Z}_3 \\ 9\mathbb{Z}_3 & \mathbb{Z}_3^\times \end{pmatrix} \times \prod_{\ell \neq 2,3 \text{ prime}} \mathrm{GL}_2(\mathbb{Z}_\ell).$$

Definition 1.9. Consider the monoid

$$\mathbf{M} := \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}^{\det \neq 0}. \tag{1.5}$$

It acts from the *right* on $\mathbb{Q}_p[x]^{\deg \leq k-2}$ by

$$h\|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) = \psi_m(d)(cz+d)^{k-2} h\Big(\frac{az+b}{cz+d}\Big).$$

We define the space of automorphic forms $S_k^D(K_0(Np^m); \psi_m)$ to be the space of functions $\varphi : (D \otimes_\mathbb{Q} \mathbb{A}_f)^\times \to \mathbb{Q}_p(\psi_m)$ such that

$$\varphi(\delta x u) = \varphi(x)\|_{u_p} \text{ for } \delta \in D^\times,\ u \in K_0(Np^m), \tag{1.6}$$

where u_p denotes the p-component of u (hence belongs to K_p).

It is clear that in each double coset $D^\times \gamma K_0(Np^m)$, if we know the value at one element, then formula (1.6) will tell us the value at all other elements.

Fact 1.10. The number of double cosets $D^\times \backslash (D \otimes_\mathbb{Q} \mathbb{A}_f)^\times / K_0(Np^m)$ is finite. When Np^m is sufficiently large, we can write

$$(D \otimes_\mathbb{Q} \mathbb{A}_f)^\times = \coprod_{i=1}^{t} D^\times \gamma_i K_0(Np^m).$$

such that for each i, the natural map

$$\begin{array}{ccc} D^\times \times K_0(Np^m) & \longrightarrow & D^\times \gamma_i K_0(Np^m) \\ (\delta, u) & \longmapsto & \delta \gamma_i u \end{array}$$

is two-to-one and sending (δ, u) and $(-\delta, -u)$ to the same element. We call this Np^m *neat*. Moreover, in the way we setup the level structure, we may always take each γ_i to satisfy the following properties:

- $\gamma_{i,p} = 1$,
- $\det(\gamma_{i,\ell}) \in \mathbb{Z}_\ell^\times$ for each prime ℓ.

For the proof of this existence, see [WXZ14$^+$, Notation 4.1].

For our particular D and $K_0(18)$, we there is only one γ (which we may take to be just 1. More precisely, we have a bijection

$$\begin{array}{ccc} D^\times \times_{\{\pm 1\}} K_0(18) & \xrightarrow{\cong} & (D \otimes_\mathbb{Q} \mathbb{A}_f)^\times \\ (\delta, u) & \longmapsto & \delta u \end{array} \tag{1.7}$$

We include the proof of this fact in the appendix (which will be important when we try to generalize our computation beyond this particular case).

Corollary 1.11. *When Np^m is neat, evaluating at the chosen coset representatives γ_i gives an isomorphism (assuming Hypothesis 1.3)*

$$\begin{array}{ccc} S_k^D(K_0(Np^m);\psi_m) & \xrightarrow{\cong} & \oplus_{i=1}^t \mathbb{Q}_p[x]^{\deg \leq k-2} \\ \varphi & \longmapsto & \varphi(\gamma_i). \end{array} \tag{1.8}$$

In view of this, the space of automorphic forms on D is much much simpler than the space of modular forms!

1.12. **Hecke actions.** To compare with classical modular forms, we need to define Hecke actions.

Let ℓ be a prime that does not divide Np; then $K_\ell \simeq \mathrm{GL}_2(\mathbb{Z}_\ell)$. We write $K_\ell \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) K_\ell = \coprod_{i=0}^{\ell} K_\ell w_i$, with $w_i = \left(\begin{smallmatrix} \ell & 0 \\ i & 1 \end{smallmatrix}\right)$ for $i = 0, \dots, \ell-1$ and $w_\ell = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right)$, viewed as elements in $\mathrm{GL}_2(\mathbb{Q}_l) \simeq D \otimes_{\mathbb{Q}} \mathbb{Q}_l$. We define the action of the operator T_ℓ on $S_k^D(K_0(Np^m);\psi_m)$ by

$$T_\ell(\varphi) = \sum_{i=0}^{l} \varphi|_{w_i}, \quad \text{with } (\varphi|_{w_i})(g) := \varphi(g w_i^{-1}).^{2}$$

At p, recall that $K_p = \left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right)$ with $m \geq 1$, so we have a decomposition

$$K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p = \coprod_{i=0}^{p-1} K_p v_i, \quad \text{with } v_i = \left(\begin{smallmatrix} p & 0 \\ ip^m & 1 \end{smallmatrix}\right). \tag{1.9}$$

Then the action of the operator U_p on $S_k^D(K_0(Np^m);\psi_m)$ is defined to be

$$U_p(\varphi) = \sum_{i=0}^{p-1} \varphi|_{v_i}, \quad \text{with } (\varphi|_{v_i})(g) := \varphi(g v_i^{-1})\|_{v_i}. \tag{1.10}$$

We point out that the definition of U_p- and T_l-operators do not depend on the choices of the double coset representatives w_i and v_i. But our choices may ease the computation.

These U_p- and T_l-operators are viewed as acting on the space on the left (although the expression seems to suggest a right action); they are pairwise commutative.

Proposition 1.13. *In terms of the explicit description of the space of overconvergent automorphic forms, the U_p- and T_ℓ- (for $\ell \nmid Np$) operators can be described by the following commutative diagram.*

$$\begin{array}{ccc} S^{D,\dagger}(U;\kappa) & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))} & \oplus_{i=0}^{t-1} A \widehat{\otimes} \mathcal{A} \\ {\scriptstyle \varphi \mapsto U_p\varphi} \big\downarrow {\scriptstyle \varphi \mapsto T_\ell\varphi} & & {\scriptstyle \mathfrak{U}_p} \big\downarrow {\scriptstyle \mathfrak{T}_\ell} \ \textit{Map of interest} \\ S^{D,\dagger}(U;\kappa) & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))} & \oplus_{i=0}^{t-1} A \widehat{\otimes} \mathcal{A}. \end{array}$$

Here the right vertical arrow $\mathfrak{U}_p$ (resp. $\mathfrak{T}_l$) is given by a matrix with the following description.

1. *The entries of $\mathfrak{U}_p$ (resp. $\mathfrak{T}_l$) are sums of operators of the form $||_{\delta_p}$, where δ_p is the p-component of a* global *element $\delta \in D^\times$* of norm p (resp. norm ℓ)*.*
2. *There are exactly p (resp. $\ell + 1$) such operators appearing in each row and each column of $\mathfrak{U}_p$ (resp. $\mathfrak{T}_\ell$).*

[2] This looks slightly different from (1.10) below because $|_{w_i}$ is trivial as w_i is not in the p-component.

(3) *Viewing the global element* $\delta \in D^\times$ *as an element of* $(D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$*, we have* $\delta \in \left(\begin{smallmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right)$ *(resp.* $\delta \in \left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right)$*).*

Proof. Set $K := K_0(Np^m)$ for simplicity. We only prove this for the U_p-operator and the proof for the T_l-operator ($l \nmid Np$) is similar. For each γ_i, we have

$$(U_p\varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\gamma_i v_j^{-1})\|_{v_j}.$$

Now we can write each $\gamma_i v_j^{-1}$ *uniquely* as $\delta_{i,j}^{-1}\gamma_{\lambda_{i,j}} u_{i,j}$ for $\delta_{i,j} \in D^\times$, $\lambda_{i,j} \in \{0, \dots, t-1\}$, and $u_{i,j} \in K$. Then we have

$$(U_p\varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\delta_{i,j}^{-1}\gamma_{\lambda_{i,j}} u_{i,j})\|_{v_j} = \sum_{j=0}^{p-1} \varphi(\gamma_{\lambda_{i,j}})\|_{u_{i,j,p}v_j},$$

where $u_{i,j,p}$ is the p-component of $u_{i,j}$. Substitute back in $u_{i,j}v_j = \gamma_{\lambda_{i,j}}^{-1}\delta_{i,j}\gamma_i$ and note the fact that both γ_i and $\gamma_{\lambda_{i,j}}$ have trivial p-component by our choice in Fact 1.10. We have

$$(U_p\varphi)(\gamma_i) = \sum_{j=0}^{p-1} \varphi(\gamma_{\lambda_{i,j}})\|_{\delta_{i,j,p}},$$

where $\delta_{i,j,p}$ is the same as the *global element* $\delta_{i,j} \in D^\times$ but viewed an element of $(D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$. We now check the description of each $\delta_{i,j}$:

$$\delta_{i,j} = \gamma_{\lambda_{i,j}} u_{i,j} v_j \gamma_i^{-1} \in \gamma_{\lambda_{i,j}} K \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K \gamma_i^{-1}.$$

From this, we see that the p-component of $\delta_{i,j}$ lies in $\left(\begin{smallmatrix} p\mathbb{Z}_p & \mathbb{Z}_p \\ p^m\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right)$. Moreover, the norm of $\gamma_{\lambda_{i,j}} K \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K \gamma_i^{-1}$ lands in $p\prod_{\ell \text{ prime}} \mathbb{Z}_\ell^\times$, because our choice of the representatives satisfies $\mathrm{Nm}(\gamma_i) \in \prod_{\ell \text{ prime}} \mathbb{Z}_\ell^\times$ by Fact 1.10. Therefore, $\mathrm{Nm}(\delta_{i,j}) \in \mathbb{Q}_{>0}^\times \cap p\prod_{\ell \text{ prime}} \mathbb{Z}_\ell^\times = \{p\}$. This concludes the proof of the proposition. ☐

Everything we developed so far is meaningful because we have the following big theorem

Theorem 1.14 (Jacquet–Langlands)**.** *There is an isomorphism*

$$S_k(\Gamma_0(Np^m); \psi_m)^{\Sigma_D\text{-new}} \cong S_k^D(K_0(Np^m); \psi_m)$$

respecting the actions of T_ℓ *and* U_p *on both sides.*

For the particular case appearing in our project, we have an isomorphism

$$S_k(\Gamma_0(18); \psi_9)^{2\text{-new}} \cong S_k^D(K_0(18); \psi_9) \tag{1.11}$$

respecting the action of U_3 *and* T_ℓ *for all* $\ell \neq 2, 3$*.*

Remark 1.15. The upshot of this project is that: the computation of modular forms is often very involved. But via the Jacquet–Langlands correspondence, we may transfer the computation to the world of definite quaternion algebra, where everything is often simpler.

2. Computation of our particular case

2.1. **Proof of the isomorphism** (1.7)**.** This is of course coincidental for our choices of D, p and $K_0(18)$. We first put $\mathcal{O}_D := \mathbb{Z}\langle \mathbf{i}, \mathbf{j}\rangle/(\mathbf{i}^2 = \mathbf{j}^2 = -1, \mathbf{ij} = -\mathbf{ji})$; it is the *integral subring* of D. One can check that $\mathcal{O}_D$ has 24 units:

$$\mathcal{O}_D^\times = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \tfrac{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}}{2}\}.$$

We consider a maximal open compact subgroup of $D^\times$:

$$K_{\max} := \mathbb{Z}_4\langle \varpi\rangle^\times \times \prod_{\ell \neq 2 \text{ prime}} \mathrm{GL}_2(\mathbb{Z}_\ell).$$

It is fact that $(D \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times = D^\times \cdot K_{\max}$ (see [Ja04, Lemma 1.22]). (This fact is special to our particular quaternion algebra, and is probably true for a couple of more quaternion algebras.)

Taking into account of the duplication, we have

$$D_f^\times = D^\times \times_{\mathcal{O}_D^\times} K_{\max}.$$

So it suffices to check that the image of $\mathcal{O}_D^\times/\{\pm 1\}$ in $\mathrm{GL}_2(\mathbb{Z}_3)/\{\pm 1\}$ turns out to form a coset representative of $K_{\max}/K_0(18)$. (At least they both have cardinality 12.) This can be checked easily by hand (as done in [Ja04, Theorem 2.1]) Indeed, we just simply list all 24 elements of $\mathcal{O}_D^\times$ modulo 9 using our chosen identification of $D \otimes_{\mathbb{Q}} \mathbb{Q}_3 \cong \mathrm{M}_2(\mathbb{Q}_3)$ in (1.1). Put

$$u_1 = \tfrac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad u_2 = \tfrac{1}{2}(-1+\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad u_3 = \tfrac{1}{2}(1-\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad u_4 = \tfrac{1}{2}(1+\mathbf{i}-\mathbf{j}+\mathbf{k}),$$

$$u_5 = \tfrac{1}{2}(1+\mathbf{i}+\mathbf{j}-\mathbf{k}), \quad u_6 = \tfrac{1}{2}(-1-\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad u_7 = \tfrac{1}{2}(-1+\mathbf{i}-\mathbf{j}+\mathbf{k}), \quad \text{and} \quad u_8 = \tfrac{1}{2}(-1+\mathbf{i}+\mathbf{j}-\mathbf{k}).$$

Then modulo 9, we have

$$\pm 1 \leftrightarrow \pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \quad \pm\mathbf{i} \leftrightarrow \pm\left(\begin{smallmatrix}4&1\\1&5\end{smallmatrix}\right), \quad \pm\mathbf{j} \leftrightarrow \pm\left(\begin{smallmatrix}0&8\\1&0\end{smallmatrix}\right), \quad \pm\mathbf{k} \leftrightarrow \pm\left(\begin{smallmatrix}1&5\\5&8\end{smallmatrix}\right),$$

$$\pm u_1 \leftrightarrow \pm\left(\begin{smallmatrix}3&7\\8&7\end{smallmatrix}\right), \quad \pm u_2 \leftrightarrow \pm\left(\begin{smallmatrix}2&7\\8&6\end{smallmatrix}\right), \quad \pm u_3 \leftrightarrow \pm\left(\begin{smallmatrix}8&6\\7&2\end{smallmatrix}\right), \quad \pm u_4 \leftrightarrow \pm\left(\begin{smallmatrix}3&8\\7&7\end{smallmatrix}\right),$$

$$\pm u_5 \leftrightarrow \pm\left(\begin{smallmatrix}2&2\\3&8\end{smallmatrix}\right), \quad \pm u_6 \leftrightarrow \pm\left(\begin{smallmatrix}7&6\\7&1\end{smallmatrix}\right), \quad \pm u_7 \leftrightarrow \pm\left(\begin{smallmatrix}2&8\\7&6\end{smallmatrix}\right), \quad \pm u_8 \leftrightarrow \pm\left(\begin{smallmatrix}1&2\\3&7\end{smallmatrix}\right).$$

Lemma 2.2. *For the case considered in this section, the map $\mathfrak{U}_3$ in Proposition 1.13 is given by $\mathfrak{U}_3 = ||_{\delta_1} + ||_{\delta_2} + ||_{\delta_3}$, where*

$$\delta_1 = \pm(-1+\mathbf{i}-\mathbf{j}), \ \delta_2 = \pm\tfrac{1}{2}(1+\mathbf{i}+3\mathbf{j}+\mathbf{k}), \ \textit{and} \ \delta_3 = \pm\tfrac{1}{2}(1-3\mathbf{i}-\mathbf{j}-\mathbf{k}).$$

The images of $\delta_1, \delta_2, \delta_3$ in $\mathrm{GL}_2(\mathbb{Z}_3)$ are given by

$$\pm\begin{pmatrix}\nu_3 - 1 & 2\\ 0 & -1-\nu_3\end{pmatrix}, \quad \pm\begin{pmatrix}1+\frac{\nu_3}{2} & -1-\frac{\nu_3}{2}\\ 2-\frac{\nu_3}{2} & -\frac{\nu_3}{2}\end{pmatrix}, \quad \pm \textit{and} \begin{pmatrix}-\frac{3\nu_3}{2} & -1+\frac{\nu_3}{2}\\ -2+\frac{\nu_3}{2} & 1+\frac{3\nu_3}{2}\end{pmatrix}.$$

Modulo 9, they are

$$\pm\begin{pmatrix}3&2\\0&4\end{pmatrix}, \quad \pm\begin{pmatrix}3&6\\0&7\end{pmatrix}, \quad \textit{and} \quad \pm\begin{pmatrix}3&1\\0&7\end{pmatrix}.$$

Proof. We follow the computation in Proposition 1.13. We need to compute

$$U_3(\varphi)(1) = \sum_{j=1}^{3} \varphi(v_j^{-1})||_{v_j}, \quad \text{for } v_j = \left(\begin{smallmatrix}3&0\\9j&1\end{smallmatrix}\right)$$

Using the bijection (1.1), we can write each v_j^{-1} uniquely as $\delta_j^{-1}u_j$ for $\delta_j \in D^\times$ and $u_j \in U$. Then

$$\varphi(v_j^{-1})\|_{v_j} = \varphi(1)\|_{u_{j,3}v_j} = \varphi(1)\|_{\delta_{j,3}},$$

where $u_{j,3}$ and $\delta_{j,3}$ denote the 3-components of u_j and δ_j, respectively. On the other hand, we have

$$\delta_j = u_j v_j \in D^\times \cap K_3 v_j \subseteq D^\times \cap K_3\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)K_3 = D^\times \cap K_p\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right).$$

If we put $\delta_j = \delta'_j(1-\mathbf{i}+\mathbf{j})$, then we have

$$\begin{aligned}\delta'_j &\in D^\times \cap K_3\left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)(1-\mathbf{i}+\mathbf{j})^{-1} = D^\times \cap K_3\left(\begin{smallmatrix}1+\nu_3&2\\0&(1-\nu_3)/3\end{smallmatrix}\right)\\ &= D^\times \cap K_3\left(\begin{smallmatrix}5&2\\0&2\end{smallmatrix}\right) = \left\{\pm 1,\ \pm\tfrac{1}{2}(1+\mathbf{i}+\mathbf{j}-\mathbf{k}),\ \pm\tfrac{1}{2}(1-\mathbf{i}-\mathbf{j}+\mathbf{k})\right\}\end{aligned}$$

The last equality follows from looking at the list of $\mathcal{O}_D^\times$ modulo 3. (In the notation above, this set is $\{\pm 1, \pm u_5, \pm u_8\}$.)

It is then clear that all δ_j's are among the collections of the above right-multiplied by $1-\mathbf{i}+\mathbf{j}$. The rest of the lemma is straightforward. □

Conclusion 2.3. The action of U_3 on $S_k(\Gamma_0(18);\psi_3)^{\text{2-new}}$ is the same as the action of the following operator on $\mathbb{Q}_3[x]^{\deg\leq k-2}$:

$$\begin{aligned}U_p(h)(z) :&= \zeta_3\cdot(-1-\nu_3)^{k-2}h\Big(\frac{(\nu_3-1)z+2}{-1-\nu_3}\Big)\\ &\quad+\zeta_3^2\big((2-\tfrac{\nu_3}{2})z-\tfrac{\nu_3}{2}\big)^{k-2}h\Big(\frac{(2+\nu_3)z-2-\nu_3}{(4-\nu_3)z-\nu_3}\Big)\\ &\quad+\zeta_3^2\big((-2-\tfrac{\nu_3}{2})z+1-\tfrac{3\nu_3}{2}\big)^{k-2}h\Big(\frac{-3\nu_3 z-2+\nu_3}{(-4+\nu_3)z+2+3\nu_3}\Big).\end{aligned}$$

Project 2.4. Realize this computation on SAGE. Do similar computation for other Hecke operators T_ℓ, and other level structure. How do we generalize this to other definite quaternion algebras?

3. p-adic family of overconvergent automorphic forms

A next-step question is to generalize the computation above to the case of p-adic families, letting the weight k to vary p-adically. But at first sight, this does not make sense as the dimension of the space $S_k^D(K_0(Np^m);\psi_m)$ changes as k changes. So the idea is to make everything infinite dimensional.

3.1. **Universal characters.** We shall only study a very special part of the overconvergent automorphic forms. For general theory, see e.g. [Bu04] or [WXZ14+].

Let p be an odd prime. Recall that $\mathbb{Z}_p^\times \cong \mu_{p-1}\times(1+p\mathbb{Z}_p)^\times$. So for each $a \in \mathbb{Z}_p^\times$, we write $[a]$ for the unique $(p-1)$st root of unity in $\mathbb{Z}_p$ that is congruent to a modulo p, and put $\langle a\rangle = a/[a] \in 1+p\mathbb{Z}_p$. Then the isomorphism above is given by sending $a \mapsto ([a], \langle a\rangle)$.

Recall that we have a p-adic logarithmic map $\log : \mathbb{Z}_p^\times \to p\mathbb{Z}_p$ given by

$$\log(a) := \log(\langle a\rangle) = 1 + \big(\langle a\rangle - 1\big) - \tfrac{1}{2}\big(\langle a\rangle - 1\big)^2 + \tfrac{1}{3}\big(\langle a\rangle-1\big)^3 + \cdots.$$

Consider the Tate algebra

$$\mathbb{Z}_p\langle T\rangle := \Big\{ \sum_{n\geq 0} a_n T^n \;\Big|\; a_n \in \mathbb{Z}_p,\ \text{and } a_n \to 0 \Big\}, \quad \text{and} \quad \mathbb{Q}_p\langle T\rangle := \mathbb{Z}_p\langle T\rangle\Big[\frac{1}{p}\Big].$$

We consider the *universal character*

$$\psi_{\mathrm{univ},m} : \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p[\zeta_{p^{m-2}}]\langle T\rangle^\times$$

$$\psi_{\mathrm{univ},m}(a) = \psi_m(a)\cdot \exp(T\cdot \log a) = \psi_m(a)\cdot (1 + T\log a + \tfrac{1}{2!}(T\log a)^2 + \cdots).$$

In particular, when specializing to $T = k \in \mathbb{Z}$, this is usual character $a \mapsto \psi_m(a)\langle a\rangle^k$.

3.2. **Overconvergent automorphic forms.** We consider the universal action of $\mathbf{M}$ (see (1.5)) on $\mathbb{Q}_p(\zeta_{p^{m-2}})\langle T\rangle\langle z\rangle$ given by

$$h||_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}(z) := \psi_{\mathrm{univ},m}(d)\exp\big(T\cdot\log(1+\tfrac{c}{d}z)\big)h\big(\tfrac{az+b}{cz+d}\big).$$

Then we define the space $S^{D,\dagger}_{\psi_{\mathrm{univ},m}}(K_0(Np^m))$ of *overconvergent automorphic forms* with weight in $\psi_{\mathrm{univ},m}$ and level Np^m to be the space of functions $\varphi : (D\otimes_{\mathbb{Q}}\mathbb{A}_f)^\times \to \mathbb{Q}_p(\zeta_{p^{m-2}})\langle T\rangle\langle z\rangle$ such that

$$\varphi(\delta x u) = \varphi(x)\|_{u_p} \text{ for } \delta\in D^\times,\ u \in K_0(Np^m), \tag{3.1}$$

Similar to Corollary 1.11, evaluating at the coset representatives γ_i gives an isomorphism

$$S^{D,\dagger}_{\psi_{\mathrm{univ},m}}(K_0(Np^m)) \xrightarrow{\ \cong\ } \bigoplus_{i=1}^{t} \mathbb{Q}_p(\zeta_{p^{m-2}})\langle T\rangle\langle z\rangle, \tag{3.2}$$

and the Hecke actions are given by the same formulas explained in Section 1.12, which translate to the action on the right hand side of (3.2) via Proposition 1.13.

Now, we wish to make the action of U_p even more explicit, by expressing the action of $||_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}$ on each direct summand $\mathbb{Q}_p(\zeta_{p^{m-2}})\langle T\rangle\langle z\rangle$ in terms of an infinite matrix with respect to the basis $1, z, z^2, \ldots$

3.3. **Infinite matrices and generating functions.** For an infinite matrix (where the row and column indices start with 0 as opposed to 1)

$$M = \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots \\ m_{2,0} & m_{2,1} & m_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.3}$$

with coefficients in a ring A, we consider the following formal power series:

$$H_M(x,y) = \sum_{i,j\in\mathbb{Z}_{\geq 0}} m_{i,j}x^iy^j \in A[\![x,y]\!].$$

It is called the *generating series* of the matrix M. Consider our case, we write $H_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)}(x,y) \in \mathbb{Q}_p(\zeta_{p^{m-1}})[\![x,y]\!]$ for the action of the matrix $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathbf{M}$ on the Tate algebra $\mathbb{Q}_p(\zeta_{p^{m-2}})\langle T\rangle\langle z\rangle$, with respect to the basis $1, z, z^2, \ldots$.

The following key calculation is due to Jacobs [Ja04, Proposition 2.6].

Proposition 3.4. *The generating series of the operator* $||_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$ *acting on* $\mathbb{Q}_p(\zeta_{p^{m-2}})\langle T\rangle\langle z\rangle$ *(with respect to the basis* $1, z, z^2, \dots$*) is given by*

$$\frac{\psi_{\mathrm{univ},m}(d)\exp(T\cdot\log(1+\frac{c}{d}x))(cx+d)}{cx+d-axy-by}.$$

Proof. This is straightforward. By definition,

$$\begin{aligned} H_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(x,y) &= \sum_{i\in\mathbb{Z}_{\geq 0}} y^i\cdot\psi_{\mathrm{univ},m}(d)\exp(T\cdot\log(1+\frac{c}{d}x))\cdot\big(\frac{ax+b}{cx+d}\big)^i \\ &= \psi_{\mathrm{univ},m}(d)\exp(T\cdot\log(1+\frac{c}{d}x))\cdot\frac{1}{1-y\cdot\frac{ax+b}{cx+d}} \\ &= \frac{\psi_{\mathrm{univ},m}(d)\exp(T\cdot\log(1+\frac{c}{d}x))(cx+d)}{cx+d-axy-by}. \end{aligned}$$ □

Combining Proposition 3.4 with Proposition 1.13, we can give a good description of the infinite matrices for $\mathfrak{U}_p$.

In our particular case, we can use this computation to give an estimate of the p-adic valuations of the U_p-action on the $S^{D,\dagger}_{\psi_{\mathrm{univ},2}}(K_0(18))$.

Theorem 3.5 (Jacobs)**.** *Evaluating* T *at* $w\in\mathcal{O}_{\mathbb{C}_3}$ *so that we are considering the character* $\kappa=\psi_2 x^w$*. The slopes of the* U_3*-operator acting on* $S^{D,\dagger}_\kappa(K_0(18))$ *are* $\frac12, 1+\frac12, 2+\frac12, 3+\frac12, \dots$*. (Note the sequence is independent of* w*.)*

Proof. Exercise. □

Project 3.6. Write a SAGE code to compute the characteristic power series of the U_p-action on $S^{D,\dagger}_{\psi_{\mathrm{univ},2}}(K_0(18))$.

Generalize this code to deal with other situations. Eventually, we hope that this code can get to efficiency close to [BP15+], and shed some light on the ghost conjecture in [BP16+].

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