TRANSFORMATION FOR FAKE MODULAR FORM OF WEIGHT 2

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1. Introduction

For even $k \geq 4$ the normalized Eisenstein series $E_k(\tau) = G_k(\tau)/(2\zeta(k))$ has q-expansion $1 - (2k/B_k)\sum_{n\geq 1}\sigma_{k-1}(n)q^n$, where $q = e^{2\pi i\tau}$ and B_k is the kth Bernoulli number. This q-expansion suggests defining an Eisenstein series for weight 2:

$$E_2(\tau) := 1 - (4/B_2)\sum_{n\geq 1}\sigma_1(n)q^n = 1 - 24\sum_{n\geq 1}\sigma_1(n)e^{2\pi i n\tau},$$

since $B_2 = 1/6$.

While $E_2(\tau)$ is holomorphic on the upper half-plane $\mathfrak{h}$, $E_2(\tau+1) = E_2(\tau)$, and $E_2(\tau) \to 1$ as $\tau \to i\infty$, E_2 does not satisfy the weight 2 modularity condition $E_2(-1/\tau) = \tau^2 E_2(\tau)$. Indeed, it can't since there aren't any (nonzero) modular forms of weight 2 for $\mathrm{SL}_2(\mathbf{Z})$. Thus $E_2(-1/\tau)$ and $\tau^2 E_2(\tau)$ can't be equal somewhere. In fact there is a systematic discrepancy: for all $\tau \in \mathfrak{h}$,

$$E_2(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau. \tag{1.1}$$

The goal of this project is to prove (1.1) by a method different from the way it is treated in textbooks, where the technique is usually rearrangement of a double series that is not absolutely convergent or a trick called Hecke summation. Instead we will derive (1.1) from the properties of a Dirichlet series naturally associated to $E_2(\tau)$.

Since both sides of (1.1) are holomorphic, to prove (1.1) it suffices to prove it on the positive imaginary axis, where (writing $\tau = iy$) it says

$$E_2(i/y) = -y^2 E_2(iy) + \frac{6}{\pi}y. \tag{1.2}$$

Our goal is to prove this formula for all $y > 0$.

2. Background

Gamma-function. The Gamma-function is defined for $\mathrm{Re}(s) > 0$ as

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t},$$

which is holomorphic in s. For $n \in \mathbf{Z}^+$, $\Gamma(n) = (n-1)!$. Using integration by parts we have $\Gamma(s+1) = s\Gamma(s)$, and this equation allows $\Gamma(s)$ to be extended step by step to the left, making it holomorphic on $\mathbf{C}$ with no zeros and with simple poles only at nonpositive integers, where the residue is given by $\mathrm{Res}_{s=-m}\Gamma(s) = (-1)^m/m!$.

One of the many identities satisfied by the Gamma-function is the duplication formula

$$\Gamma(s) = \frac{1}{2\sqrt{\pi}}2^s\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right).$$

Fourier transform/Fourier inversion. For a "nice" function $f\colon \mathbf{R} \to \mathbf{C}$, its Fourier transform is a function $\widehat{f}\colon \mathbf{R} \to \mathbf{C}$ defined by

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixy}\,dx.$$

The Fourier inversion formula says, again for nice functions, that we can express f as an integral of $\widehat{f}$:

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(y)e^{-2\pi ixy}\,dy.$$

Note the appearance of the minus sign in the exponent of the inversion formula. (There are many conventions about how to define the Fourier transform, differing in where a factor of 2π or a minus sign appear.)

Dirichlet series. These are infinite series of the form $\sum_{n\geq 1} a_n/n^s$. If a Dirichlet series converges somewhere then it converges at all s with larger real part (a right half-plane), and it is analytic on open right half-planes where it converges (analogous to a power series being analytic on any open disc where it converges).

Riemann zeta-function. The most famous Dirichlet series is the Riemann zeta-function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s},$$

which converges and is analytic for $\mathrm{Re}(s) > 1$. A famous evaluation by Euler is $\zeta(2) = \pi^2/6$. Using formulas other than its original definition, $\zeta(s)$ can be extended analytically to $\mathbf{C}$ except for a simple pole at $s = 1$ with residue 1. We have $\zeta(0) = -1/2$ and $\zeta(s)$ has simple zeros at negative even integers. All other zeros lie in the strip $0 < \mathrm{Re}(s) < 1$ and the famous Riemann hypothesis states that all zeros of $\zeta(s)$ in this strip have real part $1/2$ (you do *not* need the Riemann hypothesis for this project).

There is a *functional equation* expressing $\zeta(s)$ in terms of $\zeta(1-s)$, but it is a rather ugly equation. A nicer functional equation is satisfied by what is called the "completed" zeta-function $Z(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Its functional equation is $Z(s) = Z(1-s)$ for all $s \in \mathbf{C}$.

3. The project

The coefficients of $E_2(\tau)$, aside from its constant term $a_0 = 1$, are the numbers $-24\sigma_1(n)$. We have $\sigma_1(n) = \sum_{d|n} d = O(n^2)$.

For a sequence of numbers $a_1, a_2, a_3, \ldots$ having at most polynomial growth, meaning $a_n = O(n^r)$ for some $r > 0$, we will associate two series having these as the coefficients:

$$g(s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1} \frac{a_n}{n^s}$$

and

$$h(y) = \sum_{n\geq 1} a_n e^{-2\pi ny}.$$

The Dirichlet series $\sum a_n/n^s$ converges absolutely for $\mathrm{Re}(s) > r + 1$ (by comparing terms with those in the Riemann zeta-function) and the series $h(y)$ converges for all real $y > 0$.

1. Use the definition of $\Gamma(s)$ as an integral (for $\mathrm{Re}(s) > 0)$) to show $g(s) = \int_0^\infty h(t)t^s dt/t$ for s with real part greater than $r+1$. In this integral, make the change of variables $t = e^x$ for $x \in \mathbf{R}$ to show for each $c > r+1$ that $g(c + 2\pi iy)$ is the Fourier transform of $h(e^x)e^{cx}$.

2. Use the Fourier inversion formula to show for $c > r+1$ that

$$h(y) = \frac{1}{2\pi i}\int_{c-1\infty}^{c+i\infty} g(s)\frac{ds}{y^s}$$

for $y > 0$. The vertical integral $\int_{c-1\infty}^{c+i\infty}$ means $\lim_{T\to\infty}\int_{c-iT}^{c+iT}$.

3. Setting $h(y) = E_2(iy) - 1 = \sum_{n\geq 1} -24\sigma_1(n)e^{-2\pi ny}$, show for $\mathrm{Re}(s) > 3$ that

$$g(s) = -24(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1) = -\frac{6}{\pi}(s-1)Z(s)Z(s-1),$$

where $Z(s)$ is the completed zeta-function. Conclude from the functional equation for $Z(s)$ that $g(2-s) = -g(s)$ for all $s \in \mathbf{C}$. By part (2),

$$E_2(iy) - 1 = \frac{1}{2\pi i}\int_{4-i\infty}^{4+i\infty} g(s)\frac{ds}{y^s}.$$

4. Show $g(s)$ has only three poles and compute the residues at them (make sure to avoid sign errors).

5. Shift the contour in the integral in (3) to the line $\mathrm{Re}(s) = -2$ and use the residue theorem to deduce

$$E_2(iy) = \frac{6}{\pi y} - \frac{1}{y^2} + \frac{1}{2\pi i}\int_{-2-i\infty}^{-2+i\infty} g(s)\frac{ds}{y^s}.$$

(If you want to be careful about estimates, look up the complex Stirling's formula to see how the Gamma-function decays at numbers far from the x-axis.)

6. Use the functional equation $g(2-s) = -g(s)$ to show

$$E_2(iy) = \frac{6}{\pi y} - \frac{1}{y^2}E_2(i/y),$$

which is equivalent to (1.2).