TRANSFORMATION FOR FAKE MODULAR FORM OF WEIGHT 2

KEITH CONRAD

1. Introduction

For even \( k \geq 4 \) the normalized Eisenstein series \( E_k(\tau) = G_k(\tau)/(2\zeta(k)) \) has \( q \)-expansion
\[
1 - (2k/B_k) \sum_{n \geq 1} \sigma_k(n)q^n,
\]
where \( q = e^{2\pi i \tau} \) and \( B_k \) is the \( k \)th Bernoulli number. This \( q \)-expansion suggests defining an Eisenstein series for weight 2:
\[
E_2(\tau) := 1 - \frac{4}{B_2} \sum_{n \geq 1} \sigma_1(n)q^n = 1 - 24 \sum_{n \geq 1} \sigma_1(n)e^{2\pi in\tau},
\]
since \( B_2 = 1/6 \).

While \( E_2(\tau) \) is holomorphic on the upper half-plane \( \mathfrak{h} \), \( E_2(\tau+1) = E_2(\tau) \), and \( E_2(\tau) \to 1 \) as \( \tau \to i\infty \), \( E_2 \) does not satisfy the weight 2 modularity condition \( E_2(-1/\tau) = \tau^2 E_2(\tau) \). Indeed, it can’t since there aren’t any (nonzero) modular forms of weight 2 for \( \text{SL}_2(\mathbb{Z}) \). Thus \( E_2(-1/\tau) \) and \( \tau^2 E_2(\tau) \) can’t be equal somewhere. In fact there is a systematic discrepancy: for all \( \tau \in \mathfrak{h} \),
\[
E_2(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i}{\pi} \tau.
\]
(1.1)

The goal of this project is to prove (1.1) by a method different from the way it is treated in textbooks, where the technique is usually rearrangement of a double series that is not absolutely convergent or a trick called Hecke summation. Instead we will derive (1.1) from the properties of a Dirichlet series naturally associated to \( E_2(\tau) \).

Since both sides of (1.1) are holomorphic, to prove (1.1) it suffices to prove it on the positive imaginary axis, where (writing \( \tau = iy \)) it says
\[
E_2(i/y) = -y^2 E_2(iy) + \frac{6}{\pi} y.
\]
(1.2)
Our goal is to prove this formula for all \( y > 0 \).

2. Background

Gamma-function. The Gamma-function is defined for \( \text{Re}(s) > 0 \) as
\[
\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \frac{dt}{t},
\]
which is holomorphic in \( s \). For \( n \in \mathbb{Z}^+ \), \( \Gamma(n) = (n-1)! \). Using integration by parts we have \( \Gamma(s+1) = s\Gamma(s) \), and this equation allows \( \Gamma(s) \) to be extended step by step to the left, making it holomorphic on \( \mathbb{C} \) with no zeros and with simple poles only at nonpositive integers, where the residue is given by \( \text{Res}_{s=-m} \Gamma(s) = (-1)^m / m! \).

One of the many identities satisfied by the Gamma-function is the duplication formula
\[
\Gamma(s) = \frac{1}{2\sqrt{\pi}} 2^s \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right).\]
Fourier transform/Fourier inversion. For a “nice” function \( f : \mathbb{R} \to \mathbb{C} \), its Fourier transform is a function \( \hat{f} : \mathbb{R} \to \mathbb{C} \) defined by
\[
\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{2\pi ixy} \, dx.
\]
The Fourier inversion formula says, again for nice functions, that we can express \( f \) as an integral of \( \hat{f} \):
\[
f(x) = \int_{-\infty}^{\infty} \hat{f}(y) e^{-2\pi ixy} \, dy.
\]
Note the appearance of the minus sign in the exponent of the inversion formula. (There are many conventions about how to define the Fourier transform, differing in where a factor of \( 2\pi \) or a minus sign appear.)

Dirichlet series. These are infinite series of the form \( \sum_{n \geq 1} a_n/n^s \). If a Dirichlet series converges somewhere then it converges at all \( s \) with larger real part (a right half-plane), and it is analytic on open right half-planes where it converges (analogous to a power series being analytic on any open disc where it converges).

Riemann zeta-function. The most famous Dirichlet series is the Riemann zeta-function
\[
\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s},
\]
which converges and is analytic for \( \text{Re}(s) > 1 \). A famous evaluation by Euler is \( \zeta(2) = \pi^2/6 \). Using formulas other than its original definition, \( \zeta(s) \) can be extended analytically to \( \mathbb{C} \) except for a simple pole at \( s = 1 \) with residue 1. We have \( \zeta(0) = -1/2 \) and \( \zeta(s) \) has simple zeros at negative even integers. All other zeros lie in the strip \( 0 < \text{Re}(s) < 1 \) and the famous Riemann hypothesis states that all zeros of \( \zeta(s) \) in this strip have real part \( 1/2 \) (you do not need the Riemann hypothesis for this project).

There is a functional equation expressing \( \zeta(s) \) in terms of \( \zeta(1 - s) \), but it is a rather ugly equation. A nicer functional equation is satisfied by what is called the “completed” zeta-function \( Z(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) \). Its functional equation is \( Z(s) = Z(1 - s) \) for all \( s \in \mathbb{C} \).

3. The project

The coefficients of \( E_2(\tau) \), aside from its constant term \( a_0 = 1 \), are the numbers \(-24\sigma_1(n)\). We have \( \sigma_1(n) = \sum_{d|n} d = O(n^2) \).

For a sequence of numbers \( a_1, a_2, a_3, \ldots \) having at most polynomial growth, meaning \( a_n = O(n^r) \) for some \( r > 0 \), we will associate two series having these as the coefficients:
\[
g(s) = (2\pi)^{-s}\Gamma(s) \sum_{n \geq 1} \frac{a_n}{n^s}
\]
and
\[
h(y) = \sum_{n \geq 1} a_n e^{-2\pi ny}.
\]
The Dirichlet series \( \sum a_n/n^s \) converges absolutely for \( \text{Re}(s) > r + 1 \) (by comparing terms with those in the Riemann zeta-function) and the series \( h(y) \) converges for all real \( y > 0 \).
1. Use the definition of $\Gamma(s)$ as an integral (for $\text{Re}(s) > 0$) to show $g(s) = \int_0^\infty h(t) t^s dt / t$ for $s$ with real part greater than $r + 1$. In this integral, make the change of variables $t = e^x$ for $x \in \mathbb{R}$ to show for each $c > r + 1$ that $g(c + 2\pi iy)$ is the Fourier transform of $h(e^x)e^{cx}$.

2. Use the Fourier inversion formula to show for $c > r + 1$ that

$$h(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) \frac{ds}{y^s}$$

for $y > 0$. The vertical integral $\int_{c-i\infty}^{c+i\infty}$ means $\lim_{T \to \infty} \int_{c-iT}^{c+iT}$.

3. Setting $h(y) = E_2(iy) - 1 = \sum_{n \geq 1} -24\sigma_1(n)e^{-2\pi ny}$, show for $\text{Re}(s) > 3$ that

$$g(s) = -24(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1) = -\frac{6}{\pi}(s-1)Z(s)Z(s-1),$$

where $Z(s)$ is the completed zeta-function. Conclude from the functional equation for $Z(s)$ that $g(2-s) = -g(s)$ for all $s \in \mathbb{C}$. By part (2),

$$E_2(iy) - 1 = \frac{1}{2\pi i} \int_{4-i\infty}^{4+i\infty} g(s) \frac{ds}{y^s}.$$

4. Show $g(s)$ has only three poles and compute the residues at them (make sure to avoid sign errors).

5. Shift the contour in the integral in (3) to the line $\text{Re}(s) = -2$ and use the residue theorem to deduce

$$E_2(iy) = \frac{6}{\pi y} - \frac{1}{y^2} + \frac{1}{2\pi i} \int_{-2-i\infty}^{-2+i\infty} g(s) \frac{ds}{y^s}.$$  
(If you want to be careful about estimates, look up the complex Stirling’s formula to see how the Gamma-function decays at numbers far from the $x$-axis.)

6. Use the functional equation $g(2-s) = -g(s)$ to show

$$E_2(iy) = \frac{6}{\pi y} - \frac{1}{y^2} E_2(i/y),$$

which is equivalent to (1.2).