MODULAR FORMS (DRAFT, CTNT 2016)

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1. Introduction

A modular form is a holomorphic function on the upper half-plane

$$\mathfrak{h} = \{x + iy : x \in \mathbf{R}, y > 0\} = \{\tau \in \mathbf{C} : \operatorname{Im} \tau > 0\}$$

that transforms in a certain way under a discrete matrix group and has a nice behavior at infinity. To explain this more precisely (see Definition 1.2 below) we introduce a few 2×2 real matrix groups.

Definition 1.1. Set

$$\begin{aligned} \mathrm{GL}_2(\mathbf{R}) &= \{A \in \mathrm{M}_2(\mathbf{R}) : \det A \neq 0\}, \\ \mathrm{GL}_2^+(\mathbf{R}) &= \{A \in \mathrm{M}_2(\mathbf{R}) : \det A > 0\}, \\ \mathrm{SL}_2(\mathbf{R}) &= \{A \in \mathrm{M}_2(\mathbf{R}) : \det A = 1\}. \end{aligned}$$

These are all groups under matrix multiplication, with identity $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The notations GL and SL stand for "general linear" and "special linear," where the word "special" is shorthand for "determinant 1." Clearly $\mathrm{GL}_2(\mathbf{R}) \supset \mathrm{GL}_2^+(\mathbf{R}) \supset \mathrm{SL}_2(\mathbf{R})$.

We will be interested in discrete subgroups of $\mathrm{GL}_2(\mathbf{R})$, especially the integer-matrix analogue of $\mathrm{SL}_2(\mathbf{R})$, which is[1]

$$\mathrm{SL}_2(\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbf{Z}) : ad - bc = 1 \right\}.$$

If you pick three integers in a 2×2 matrix and solve for the fourth to have $ad - bc = 1$, usually it won't be an integer so you don't get a matrix in $\mathrm{SL}_2(\mathbf{Z})$. To create a matrix in $\mathrm{SL}_2(\mathbf{Z})$ "randomly," pick any pair of relatively prime integers for the first column and solve for the second column using Euclid's algorithm. For example, to find a matrix $\left(\begin{smallmatrix} 18 & x \\ 25 & y \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$ is the same as solving $18y - 25x = 1$ in integers x and y.

Definition 1.2. Let $k \in \mathbf{Z}$. A *modular form of weight k* for $\mathrm{SL}_2(\mathbf{Z})$ is a function $f \colon \mathfrak{h} \to \mathbf{C}$ such that

(1) f is holomorphic on $\mathfrak{h}$,
(2) $f\left(\dfrac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ and all $\tau \in \mathfrak{h}$,
(3) the values $f(\tau)$ are bounded as $\operatorname{Im} \tau \to \infty$.

Often "$\operatorname{Im} \tau \to \infty$" is written as $\tau \to i\infty$ and we think of $i\infty$ as a point infinitely high up in $\mathfrak{h}$, analogous to ∞ and $-\infty$ lying infinitely far to the right or left of $\mathbf{R}$.

[1] The group $\mathrm{GL}_2(\mathbf{Z})$ is *not* the 2×2 integer matrices with nonzero determinant, since that is not a group: the inverse of such a matrix need not have integer entries. Instead, $\mathrm{GL}_2(\mathbf{Z}) = \{A \in \mathrm{M}_2(\mathbf{Z}) : \det A = \pm 1\}$.

Remark 1.3. The three defining properties of a modular form are independent of each other: there are functions $\mathfrak{h} \to \mathbf{C}$ satisfying any two of the three properties but not satisfying the third (for some choice of k).

The zero function on $\mathfrak{h}$ is a modular form of every weight. We will eventually see that the only modular form of negative weight, odd weight, or weight 2 for $\mathrm{SL}_2(\mathbf{Z})$ is the function 0, the only modular forms of weight 0 for $\mathrm{SL}_2(\mathbf{Z})$ are constant functions, and for every even $k \geq 4$ we'll use a construction called Eisenstein series in Section 4 to give a nonzero example of a modular form of weight k for $\mathrm{SL}_2(\mathbf{Z})$.

The second property in the definition of a modular form is called the *modularity condition.* Let's make it explicit in three examples.

Example 1.4. For the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, the modularity condition means $f(\tau+1) = f(\tau)$ for all $\tau \in \mathfrak{h}$. The weight k plays no role here.

Example 1.5. For the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, the modularity condition means $f(-1/\tau) = \tau^k f(\tau)$ for all $\tau \in \mathfrak{h}$. Here we see k appears prominently.

Example 1.6. For the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, the modularity condition means $f(\tau) = (-1)^k f(\tau)$ for all $\tau \in \mathfrak{h}$, so if k is odd then f is identically zero: the only modular form of any odd weight for $\mathrm{SL}_2(\mathbf{Z})$ is the zero function.[2]

It is no surprise that modular forms might have (and do have!) applications in complex analysis, since by definition they are certain holomorphic functions. They are also connected to many other areas of math, such as combinatorics, number theory, geometry (both hyperbolic geometry and algebraic geometry), representation theory, and mathematical physics. Here are some reasons for these other connections.

(1) Modular forms can be expanded into power series in the complex variable $q = e^{2\pi i\tau}$ (this is called a q-expansion), and many q-series in combinatorics turn out to be modular forms or closely related to modular forms.
(2) The theta-function of a positive-definite quadratic form in number theory is a modular form and the L-function of an elliptic curve over $\mathbf{Q}$ (a generalization of the Riemann zeta-function) is also the L-function of a modular form. The link between elliptic curves and modular forms is how Wiles proved Fermat's Last Theorem: a counterexample to Fermat's Last Theorem leads to a contradiction of what we know about modular forms.
(3) The upper half-plane $\mathfrak{h}$ is a model for hyperbolic geometry, and constructions on $\mathfrak{h}$ that are relevant to modular forms (*e.g.*, fundamental domains and the Petersson inner product) have an appealing interpretation using the language of hyperbolic geometry.
(4) Modular forms provide embeddings of certain algebraic varieties into projective space.
(5) A modular form can be turned into a representation of an adelic matrix group.
(6) Generating functions in string theory and conformal field theory can be described in terms of modular forms.

[2]Modular forms can be defined for finite-index subgroups of $\mathrm{SL}_2(\mathbf{Z})$, and when the subgroup does not contain $-I_2$ there might be nonzero modular forms of odd weight for that subgroup.

2. Why the modularity condition?

Why would anyone think the equation

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

in the definition of a modular form is interesting? It arose from 19th century developments in complex analysis and geometry, which we will discuss in this section.

While the group $\mathrm{GL}_2(\mathbf{R})$ acts on $\mathbf{R}^2$ by linear transformations (any 2×2 matrix A sends each vector $\mathbf{v}$ in $\mathbf{R}^2$ to the vector $A\mathbf{v}$ in $\mathbf{R}^2$, and $I_2\mathbf{v} = \mathbf{v}$ and $A(B\mathbf{v}) = (AB)\mathbf{v}$ for all A and B in $\mathrm{GL}_2(\mathbf{R})$), the group $\mathrm{GL}_2^+(\mathbf{R})$ acts on $\mathfrak{h}$ by linear fractional transformations: for $\tau \in \mathfrak{h}$, define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau := \frac{a\tau+b}{c\tau+d}. \tag{2.1}$$

The reason (2.1) lies in $\mathfrak{h}$ follows from the imaginary part formula

$$\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(ad-bc)\,\mathrm{Im}\,\tau}{|c\tau+d|^2}, \tag{2.2}$$

for $\tau \in \mathbf{C} - \{-d/c\}$ and real a,b,c,d. By this formula, which the reader can check as an exercise, if $\tau \in \mathfrak{h}$ and $ad-bc > 0$ then $(a\tau+b)/(c\tau+d) \in \mathfrak{h}$. To show (2.1) defines a (left) group action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\mathfrak{h}$, check that $I_2\tau = \tau$ and $A(B\tau) = (AB)\tau$ for all A and B in $\mathrm{GL}_2^+(\mathbf{R})$.

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{R})$ and $x \in \mathbf{R}^\times$, the matrix $\left(\begin{smallmatrix} xa & xb \\ xc & xd \end{smallmatrix}\right)$ is in $\mathrm{GL}_2^+(\mathbf{R})$ (its determinant is $x^2(ad-bc)$) and it acts on $\mathfrak{h}$ in the same way as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ does since $(xa\tau+xb)/(xc\tau+xd) = (a\tau+b)/(c\tau+d)$. This is different from $\mathrm{GL}_2(\mathbf{R})$ acting as linear transformations on $\mathbf{R}^2$, where different matrices have different effects somewhere (in fact on either $\binom{1}{0}$ or $\binom{0}{1}$). Using $x = 1/\sqrt{ad-bc}$ shows every matrix in $\mathrm{GL}_2^+(\mathbf{R})$ acts on $\mathfrak{h}$ in the same way as a matrix in $\mathrm{SL}_2(\mathbf{R})$.

One of the reasons for interest in linear fractional transformations of $\mathfrak{h}$ by matrices in $\mathrm{SL}_2(\mathbf{R})$ is the classification of compact surfaces. Aside from the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C}\cup\{\infty\}$ and a torus $\mathbf{C}/L$ for any lattice L in $\mathbf{C}$, every other compact *orientable* surface can be realized as a quotient space $\Gamma\backslash\mathfrak{h} = \{\Gamma\tau : \tau \in \mathfrak{h}\}$ where $\mathfrak{h}$ is acted on from the left by some discrete subgroup Γ of $\mathrm{SL}_2(\mathbf{R})$ using linear fractional transformations. This should be thought of as a two-dimensional analogue of the construction of a circle as a quotient space $\mathbf{R}/\mathbf{Z}$, where $\mathbf{Z}$ acts on $\mathbf{R}$ as discrete additive translations ($x \mapsto x+n$ for $n \in \mathbf{Z}$).[3]

The similarity between a quotient of $\mathbf{C}$ by a lattice and a quotient of $\mathfrak{h}$ by a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ becomes more striking when we use the language of geometry: a lattice in $\mathbf{C}$ acts on $\mathbf{C}$ as a discrete group of additive translations that each preserve Euclidean distances on $\mathbf{C}$, while linear fractional transformations of $\mathfrak{h}$ coming from matrices in $\mathrm{SL}_2(\mathbf{R})$ each preserve non-Euclidean distances on $\mathfrak{h}$ when we view $\mathfrak{h}$ as the hyperbolic plane (see Appendix A). From the viewpoint of Euclidean and non-Euclidean geometry, compact orientable surfaces other than $\widehat{\mathbf{C}}$ have similar descriptions: they arise as a model geometric

[3] While $\mathbf{R}$ has a group structure, with $\mathbf{Z}$ a subgroup of $\mathbf{R}$, $\mathfrak{h}$ does not have a group structure and discrete subgroups of $\mathrm{SL}_2(\mathbf{R})$ are generally noncommutative, so we write $\Gamma\backslash\mathfrak{h}$ rather than $\mathfrak{h}/\Gamma$ to emphasize the leftness of the group action. In contrast, there is no real difference between $\mathbf{R}/\mathbf{Z}$ and $\mathbf{Z}\backslash\mathbf{R}$ since the group structure on $\mathbf{R}$ is commutative. The backslash $\backslash$ in $\mathbf{Z}\backslash\mathbf{R}$ is important since writing $\mathbf{Z}/\mathbf{R}$ would be terrible.

space ($\mathbf{C}$ or $\mathfrak{h}$) modulo the action of an appropriate[4] discrete group of distance-preserving transformations of that space.

An important way to study a space is to study nice functions (continuous, smooth, analytic) on the space. For a discrete group Γ in $\mathrm{SL}_2(\mathbf{R})$, creating nice nonconstant complex-valued functions on $\Gamma\backslash\mathfrak{h}$ is the same thing as creating nice functions $f\colon \mathfrak{h} \to \mathbf{C}$ that are Γ-invariant: $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \Gamma$ and $\tau \in \mathfrak{h}$. Two non-invariant functions lead to an invariant function if they fail to be invariant by the same fudge factor: if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \text{and} \quad g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k g(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathfrak{h}$, and the same "weight" k, then the ratio $f(\tau)/g(\tau)$ is Γ-invariant:

$$\frac{f((a\tau+b)/(c\tau+d))}{g((a\tau+b)/(c\tau+d))} = \frac{(c\tau+d)^k f(\tau)}{(c\tau+d)^k g(\tau)} = \frac{f(\tau)}{g(\tau)}.$$

But why should we use fudge factors of the form $(c\tau+d)^k$?

Suppose for a function $f\colon \mathfrak{h} \to \mathbf{C}$ that $f(\gamma\tau)$ and $f(\tau)$ are always related by a factor determined by $\gamma \in \Gamma$ and $\tau \in \mathfrak{h}$:

$$f(\gamma\tau) = j(\gamma,\tau)f(\tau) \tag{2.3}$$

for some function $j\colon \Gamma \times \mathfrak{h} \to \mathbf{C}$. That (2.1) defines a (left) group action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathfrak{h}$ means in part that $(\gamma_1\gamma_2)\tau = \gamma_1(\gamma_2\tau)$, so $f((\gamma_1\gamma_2)\tau) = f(\gamma_1(\gamma_2\tau))$. This turns (2.3) into

$$j(\gamma_1\gamma_2,\tau)f(\tau) = j(\gamma_1,\gamma_2\tau)f(\gamma_2\tau). \tag{2.4}$$

Since $f(\gamma_2\tau) = j(\gamma_2,\tau)f(\tau)$, (2.4) holds if

$$j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau), \tag{2.5}$$

which looks like the chain rule $(f_1 \circ f_2)'(x) = f_1'(f_2(x))f_2'(x)$. This suggests a natural example of (2.5) using differentiation: when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ set

$$j(\gamma,\tau) := \left(\frac{a\tau+b}{c\tau+d}\right)' = \frac{a(c\tau+d)-c(a\tau+b)}{(c\tau+d)^2} = \frac{ad-bc}{(c\tau+d)^2},$$

and for $\gamma \in \mathrm{SL}_2(\mathbf{R})$ this says $j(\gamma,\tau) = 1/(c\tau+d)^2$. When $j(\gamma,\tau)$ fits (2.5) so does $j(\gamma,\tau)^m$ for each $m \in \mathbf{Z}$, which motivates the consideration of the modularity condition with factors $1/(c\tau+d)^k$, at least for even k.

Exercises.

1. Prove (2.2).
2. Prove (2.1) defines a (left) group action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\mathfrak{h}$.
3. Prove two matrices in $\mathrm{GL}_2^+(\mathbf{R})$ act in the same way everywhere on $\mathfrak{h}$ if and only if they are scalar multiples of each other.

[4] For some discrete subgroups Γ of $\mathrm{SL}_2(\mathbf{R})$, $\Gamma\backslash\,\mathrm{SL}_2(\mathbf{R})$ is not compact.

3. Simplifying the modularity condition for $\mathrm{SL}_2(\mathbf{Z})$

The only modular forms we have seen are boring: the zero function in any weight and constant functions in weight 0. Before giving interesting example of modular forms will use group theory to simplify the modularity condition in the definition of a modular form. It is an infinite set of equations, one for each matrix in $\mathrm{SL}_2(\mathbf{Z})$, but the following lemma will let us check the modularity condition on a set of generators for $\mathrm{SL}_2(\mathbf{Z})$ to know it holds for all matrices in the group.

Lemma 3.1. *If a function $f\colon \mathfrak{h} \to \mathbf{C}$ satisfies the modularity condition with weight k for two matrices γ_1 and γ_2 in $\mathrm{SL}_2(\mathbf{Z})$ then it satisfies the modularity condition with weight k for $\gamma_1\gamma_2$ and for the inverse γ_1^{-1}.*

Proof. Let $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$. The modularity condition with weight k for these matrices says $f(\gamma_1\tau) = (c_1\tau + d_1)^k f(\tau)$ and $f(\gamma_2\tau) = (c_2\tau + d_2)^k f(\tau)$ for all $\tau \in \mathfrak{h}$. It follows that for all τ,

$$\begin{aligned} f((\gamma_1\gamma_2)\tau) &= f(\gamma_1(\gamma_2\tau)) \\ &= (c_1\gamma_2\tau + d_1)^k f(\gamma_2\tau) \\ &= (c_1\gamma_2\tau + d_1)^k (c_2\tau + d)^k f(\tau). \end{aligned}$$

Since $\gamma_2\tau = (a_2\tau + b_2)/(c_2\tau + d_2)$, a calculation shows

$$(c_1\gamma_2\tau + d_1)^k (c_2\tau + d)^k = ((c_1a_2 + d_1c_2)\tau + (c_1b_2 + d_1d_2))^k,$$

so

$$f((\gamma_1\gamma_2)\tau) = ((c_1a_2 + d_1c_2)\tau + (c_1b_2 + d_1d_2))^k f(\tau), \tag{3.1}$$

and the bottom matrix entries of

$$\gamma_1\gamma_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} * & * \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

are exactly the "c" and "d" that appear when we write $f((\gamma_1\gamma_2)\tau)$ as $(c\tau + d)^k f(\tau)$ in (3.1). Thus f satisfies the modularity condition with weight k for $\gamma_1\gamma_2$.

We now want to prove that if $f(\gamma_1\tau) = (c_1\tau + d_1)^k f(\tau)$ for all $\tau \in \mathfrak{h}$ then the same condition holds with γ_1 replaced by γ_1^{-1}, which is $\left(\begin{smallmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{smallmatrix}\right)$ because γ_1 has determinant 1. Replacing τ with $\gamma_1^{-1}\tau$ in the modularity condition for the matrix γ_1, we get

$$f(\tau) = (c_1(\gamma_1^{-1}\tau) + d_1)^k f(\gamma_1^{-1}\tau)$$

for all τ. Dividing both sides by $(c_1(\gamma_1^{-1}\tau) + d_1)^k$,

$$f(\gamma_1^{-1}\tau) = \frac{1}{(c_1\gamma_1^{-1}\tau + d_1)^k} f(\tau)$$

for all τ. Since $c_1\gamma_1^{-1}\tau + d_1 = (a_1d_1 - b_1c_1)/(-c_1\tau + a_1) = 1/(-c_1\tau + a_1)$,

$$f(\gamma_1^{-1}\tau) = (-c_1\tau + a_1)^k f(\tau)$$

for all τ, which is the modularity condition for γ_1^{-1}. □

Theorem 3.2. *If the set $\{\gamma_1, \ldots, \gamma_m\}$ generates $\mathrm{SL}_2(\mathbf{Z})$ and a function $f\colon \mathfrak{h} \to \mathbf{C}$ satisfies the modularity condition with weight k for each γ_i then f satisfies the modularity condition with weight k for all of $\mathrm{SL}_2(\mathbf{Z})$.*

Proof. By Lemma 3.1, the set of all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ for which f satisfies the modularity condition with weight k is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ (clearly the modularity condition holds when $\gamma = I_2$). Therefore if this subset contains a set of generators of $\mathrm{SL}_2(\mathbf{Z})$ it is all of $\mathrm{SL}_2(\mathbf{Z})$. □

Two particular elements in $\mathrm{SL}_2(\mathbf{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The matrix S has order 4 (check $S^2 = -I_2$), while the matrix T has infinite order (check $T^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$). As linear fractional transformations of $\mathfrak{h}$,

$$S\tau = -\frac{1}{\tau}, \quad T\tau = \tau + 1, \tag{3.2}$$

so as a transformation of $\mathfrak{h}$ the order of S is 2 rather than 4, while T has infinite order on $\mathfrak{h}$.

Theorem 3.3. *The group* $\mathrm{SL}_2(\mathbf{Z})$ *is generated by* S *and* T*.*

Proof. Let $G = \langle S, T \rangle$ be the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ generated by S and T. We will give two proofs that $G = \mathrm{SL}_2(\mathbf{Z})$, one algebraic and the other geometric.

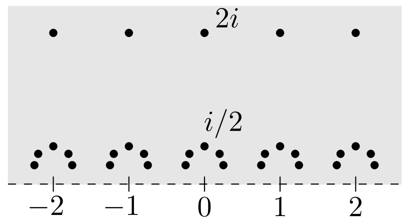
For the algebraic proof, we start by writing down the effect of S and T^n on any matrix by multiplication from the left:

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}. \tag{3.3}$$

Now pick any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$. Suppose $c \neq 0$. If $|a| \geq |c|$, divide a by c: $a = cq + r$ with $0 \leq r < |c|$. By (3.3), $T^{-q}\gamma$ has upper left entry $a - qc = r$, which is smaller in absolute value than the lower left entry c in $T^{-q}\gamma$. Applying S switches these entries (with a sign change), and we can apply the division algorithm in $\mathbf{Z}$ again if the lower left entry is nonzero in order to find another power of T to multiply by on the left so the lower left entry has smaller absolute value than before. Eventually multiplication of γ on the left by enough copies of S and powers of T gives a matrix in $\mathrm{SL}_2(\mathbf{Z})$ with lower left entry 0. Such a matrix, since it is integral with determinant 1, has the form $\left(\begin{smallmatrix} \pm 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $m \in \mathbf{Z}$ and common signs on the diagonal. This matrix is either T^m or $-T^{-m}$, so there is some $g \in G$ such that $g\gamma = \pm T^n$ for some $n \in \mathbf{Z}$. Since $T^n \in G$ and $S^2 = -I_2$, we have $\gamma = \pm g^{-1}T^n \in G$, so we are done.

In this algebraic proof, G acted on the set $\mathrm{SL}_2(\mathbf{Z})$ by left multiplication. For the geometric proof, we make G act on $\mathfrak{h}$ by linear fractional transformations. This action does not distinguish between matrices that differ by a sign (γ and $-\gamma$ act on $\mathfrak{h}$ in the same way), but this will not be a problem for the purpose of using this action to prove $G = \mathrm{SL}_2(\mathbf{Z})$ since $-I_2 = S^2 \in G$.

The key geometric idea is that when $\mathrm{SL}_2(\mathbf{Z})$ acts on a point in $\mathfrak{h}$, the orbit appears to accumulate towards the x-axis. This is illustrated by the picture below, which shows points in the $\mathrm{SL}_2(\mathbf{Z})$-orbit of $2i$ (including $S(2i) = -1/(2i) = i/2$). It appears that the imaginary parts of points in the orbit never exceed 2.



With the picture in mind, pick $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ and set $\tau := \gamma(2i)$.

For any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in G, so $ad - bc = 1$, (2.2) tells us

$$\mathrm{Im}(g\tau) = \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2}.$$

Write τ as $x + yi$. Then in the denominator

$$|c\tau + d|^2 = (cx + d)^2 + (cy)^2,$$

since $y \neq 0$ there are only finitely many integers c and d with $|c\tau + d|$ less than a given bound. Here τ is not changing but c and d are. Therefore $\mathrm{Im}(g\tau)$ has a *maximum* possible value as g runs over G (with τ fixed), so there is some $g_0 \in G$ such that $\boxed{\mathrm{Im}(g\tau) \leq \mathrm{Im}(g_0\tau)}$ for all $g \in G$.

Since $Sg_0 \in G$, the maximality property defining g_0 implies $\mathrm{Im}((Sg_0)\tau) \leq \mathrm{Im}(g_0\tau)$, so (2.2) with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = S$ gives us

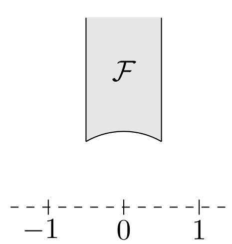
$$\mathrm{Im}(S(g_0\tau)) = \frac{\mathrm{Im}(g_0\tau)}{|g_0\tau|^2} \leq \mathrm{Im}(g_0\tau).$$

Therefore $|g_0\tau|^2 \geq 1$, so $|g_0\tau| \geq 1$. Since $\mathrm{Im}(T^n g_0\tau) = \mathrm{Im}(g_0\tau)$ and $T^n g_0 \in G$, replacing $g_0\tau$ with $T^n g_0\tau$ and running through the argument again shows $|T^n g_0\tau| \geq 1$ for all $n \in \mathbf{Z}$.

Applying T (or T^{-1}) to $g_0\tau$ adjusts its real part by 1 (or -1) without affecting the imaginary part. Every real number is in an interval $[n - 1/2, n + 1/2]$ (centered at some integer n), and if $n - 1/2 \leq \mathrm{Re}(g_0\tau) \leq n + 1/2$ then $-1/2 \leq \mathrm{Re}(T^{-n}g_0\tau) \leq 1/2$. Since $T^{-n}g_0 \in G$, the G-orbit of $\tau = \gamma(2i)$ has an element in the set

$$\mathcal{F} = \{\tau \in \mathfrak{h} : |\,\mathrm{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}. \tag{3.4}$$

See the picture below. Note $\mathrm{Im}\,\tau \geq \sqrt{3}/2 > 1/2$ for all $\tau \in \mathcal{F}$.



We started by picking the number $2i$ in $\mathcal{F}$ and any γ in $\mathrm{SL}_2(\mathbf{Z})$, and we showed there is some $g \in G$ such that the point $g(\gamma(2i)) = (g\gamma)(2i)$ is also in $\mathcal{F}$. By (2.2),

$$g\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \Longrightarrow \mathrm{Im}((g\gamma)(2i)) = \frac{2}{4c^2 + d^2} \geq \frac{\sqrt{3}}{2},$$

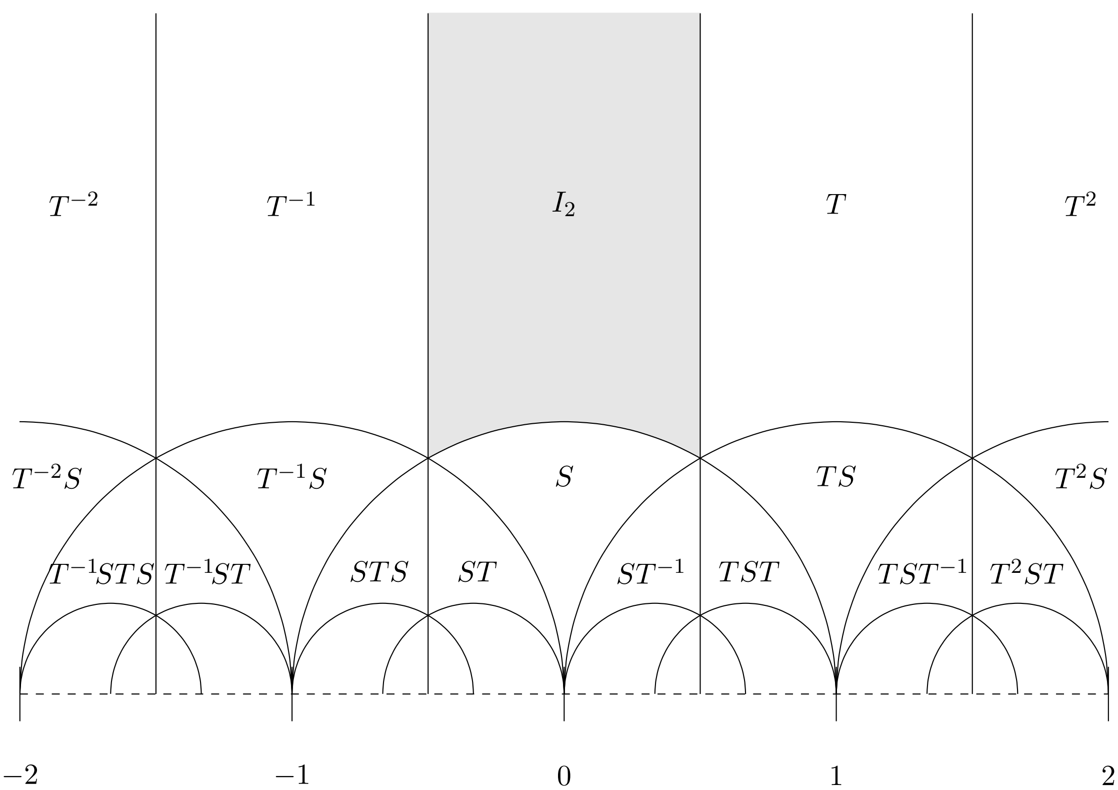
so $c = 0$ (otherwise the imaginary part is at most $2/(4c^2) \le 1/2 < \sqrt{3}/2$). Then $ad = 1$, so $a = d = \pm 1$ and

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(2i) = \frac{2ai + b}{d} = 2i \pm b.$$

For this to have real part between $\pm 1/2$ forces $b = 0$, so $g\gamma = \pm I_2$. Thus $\gamma = \pm g^{-1}$. Since $-I_2 = S^2 \in G$, we conclude $\gamma \in G$. □

The region $\mathcal{F}$ above is called a *fundamental domain* for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{h}$. It is analogous to $[0, 1]$ as a fundamental domain for the translation action of $\mathbf{Z}$ on $\mathbf{R}$: each point in the space ($\mathfrak{h}$ or $\mathbf{R}$) has a point of its orbit (by $\mathrm{SL}_2(\mathbf{Z})$ or $\mathbf{Z}$) in the fundamental domain ($\mathcal{F}$ or $[0, 1]$) and points in the fundamental domain that lie in the same orbit are on the boundary.

Below is a decomposition of $\mathfrak{h}$ into translates $\gamma(\mathcal{F})$ as γ runs over $\mathrm{SL}_2(\mathbf{Z})$, with $\gamma = I_2$ corresponding to $\mathcal{F}$. Different translates overlap only along boundary curves, and as we get closer to the x-axis $\mathfrak{h}$ is filled by infinitely many more of these translates. The fundamental domain and its translates are called "ideal triangles" since they are each bounded by three sides and have two endpoints in $\mathfrak{h}$ but one "endpoint" not in $\mathfrak{h}$: the third endpoint is either a rational number on the x-axis or (for the regions $T^n(\mathcal{F})$ with $n \in \mathbf{Z}$) is $i\infty$. The page https://roywilliams.github.io/play/js/sl2z/ animates $\mathrm{SL}_2(\mathbf{Z})$-orbits on this figure.



The description of $\mathcal{F}$ in (3.4) uses Euclidean geometry (the absolute value measures Euclidean distances in $\mathfrak{h}$) and is somewhat awkward. If we treat $\mathfrak{h}$ as the hyperbolic plane, for which the action of $\mathrm{SL}_2(\mathbf{Z})$ and more generally $\mathrm{SL}_2(\mathbf{R})$ is by isometries for the hyperbolic metric d_H (see Appendix A), then there is a prettier description of $\mathcal{F}$:

$$\mathcal{F} = \{\tau \in \mathfrak{h} : d_H(\tau, 2i) \leq d_H(\tau, \gamma(2i)) \text{ for all } \gamma \in \mathrm{SL}_2(\mathbf{Z})\}.$$

That is, $\mathcal{F}$ is the points of $\mathfrak{h}$ whose distance (as measured by the hyperbolic metric) to $2i$ is minimal compared to the distance to all points in the $\mathrm{SL}_2(\mathbf{Z})$-orbit of $2i$. The boundary of $\mathcal{F}$ is the points equidistant (for the hyperbolic metric) between $2i$ and one of its nearest $\mathrm{SL}_2(\mathbf{Z})$ translates $T(2i) = 2i+1$, $T^{-1}(2i) = 2i-1$, or $S(2i) = i/2$.[5] Part of what makes this geometric description of $\mathcal{F}$, called a *Dirichlet polygon*, attractive is that it also works for discrete groups actings by isometries on Euclidean spaces. For example, when $\mathbf{Z}$ acts on $\mathbf{R}$ by integer translations, for any $a \in \mathbf{R}$ the numbers whose distance to $a+\mathbf{Z} = \{a+n : n \in \mathbf{Z}\}$ is minimal at a is $[a - 1/2, a + 1/2]$ and this is a fundamental domain for $\mathbf{Z}$ acting on $\mathbf{R}$.

Example 3.4. We will carry out the algebraic proof of Theorem 3.3 to express $A = \left(\begin{smallmatrix} 17 & 29 \\ 7 & 12 \end{smallmatrix}\right)$ in terms of S and T.

Since $17 = 7 \cdot 2 + 3$, we want to subtract $7 \cdot 2$ from 17:

$$T^{-2}A = \begin{pmatrix} 3 & 5 \\ 7 & 12 \end{pmatrix}.$$

Now we want to switch the roles of 3 and 7. Multiply by S:

$$ST^{-2}A = \begin{pmatrix} -7 & -12 \\ 3 & 5 \end{pmatrix}.$$

Dividing -7 by 3, we have $-7 = 3 \cdot (-3) + 2$, so we want to add $3 \cdot 3$ to -7. Multiply by T^3:

$$T^3ST^{-2}A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}.$$

Once again, multiply by S to switch the entries of the first column (up to sign):

$$ST^3ST^{-2}A = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}.$$

Since $-3 = 2(-2) + 1$, we compute

$$T^2ST^3ST^{-2}A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

Mutliply by S:

$$ST^2ST^3ST^{-2}A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}.$$

Since $-2 = 1(-2) + 0$, multiply by T^2:

$$T^2ST^2ST^3ST^{-2}A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Multiply by S:

$$ST^2ST^2ST^3ST^{-2}A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = -T = S^2T.$$

Solving for A,

$$\begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} = A = T^2S^{-1}T^{-3}S^{-1}T^{-2}S^{-1}T^{-2}S^{-1}(S^2T) = T^2ST^{-3}ST^{-2}ST^{-2}ST \tag{3.5}$$

[5]We can replace $2i$ by yi for any $y > 1$ and the same description of $\mathcal{F}$ works.

since $S^{-1} = -S$.

Remark 3.5. Multiplication by the matrices S and T is closely related to continued fractions for rational numbers, with the caveat that the continued fraction algorithm should use nearest integers from above rather than from below. To illustrate, the matrix $\left(\begin{smallmatrix} 17 & 29 \\ 7 & 12 \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbf{Z})$, and to obtain an expression for it in terms of S and T, we look at the ratio of the numbers in the first column, $17/7$:

$$\frac{17}{7} = 3 - \frac{4}{7} = 3 - \frac{1}{7/4} = 3 - \frac{1}{2 - 1/4}.$$

Using the entries 3, 2, and 4 as exponents for T,

$$T^3ST^2ST^4S = \begin{pmatrix} 17 & -5 \\ 7 & -2 \end{pmatrix},$$

whose first column is what we are after. To get the correct second column, we solve $\left(\begin{smallmatrix} 17 & 29 \\ 7 & 12 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 17 & -5 \\ 7 & -2 \end{smallmatrix}\right)M$ for M, which is $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) = T^2$, so

$$\begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} = \begin{pmatrix} 17 & -5 \\ 7 & -2 \end{pmatrix} T^2 = T^3ST^2ST^4ST^2.$$

This is a different expression for $\left(\begin{smallmatrix} 17 & 29 \\ 7 & 12 \end{smallmatrix}\right)$ than the one we found in (3.5). The representation of an element of $\mathrm{SL}_2(\mathbf{Z})$ as a product of powers of S and T is not unique.

Here, finally, is the simplified description of the modularity condition in the definition of a modular form for $\mathrm{SL}_2(\mathbf{Z})$.

Corollary 3.6. *For $k \in \mathbf{Z}$, a function $f\colon \mathfrak{h} \to \mathbf{C}$ is a modular form of weight k for $\mathrm{SL}_2(\mathbf{Z})$ if and only if*

(1) *f is holomorphic on $\mathfrak{h}$,*
(2) *$f(\tau + 1) = f(\tau)$ and $f\left(-\dfrac{1}{\tau}\right) = \tau^k f(\tau)$ for all $\tau \in \mathfrak{h}$,*
(3) *the values $f(\tau)$ are bounded as $\operatorname{Im} \tau \to \infty$.*

Proof. Use Theorems 3.2 and 3.3 together with (3.2). ∎

Exercises.

1. Find a matrix in $\mathrm{SL}_2(\mathbf{Z})$ with first column $\binom{39}{14}$.
2. Express the matrix $\left(\begin{smallmatrix} 8 & 7 \\ 9 & 8 \end{smallmatrix}\right)$, which is in $\mathrm{SL}_2(\mathbf{Z})$, as a product of powers of the matrices S and T.
3. If $f\colon \mathfrak{h} \to \mathbf{C}$ is a function satisfying the modularity condition for weight 4, show $f(\omega) = 0$ where $\omega = -1/2 + i\sqrt{3}/2$ is a nontrivial cube root of unity in $\mathbf{C}$, and if instead f satisfies the modularity condition for weight 6 then prove $f(i) = 0$.
4. For $k \in \mathbf{Z}$, a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2^+(\mathbf{R})$, and a function $f\colon \mathfrak{h} \to \mathbf{C}$, define the function $f|_k\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\colon \mathfrak{h} \to \mathbf{C}$ by the formula
$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau) = \frac{1}{(c\tau + d)^k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$
   (a) Prove this formula defines a (right) group action of $\mathrm{GL}_2^+(\mathbf{R})$ on functions: $f|_k I_2 = f$ and $(f|_k A)|_k B = f|_k(AB)$ for all A and B in $\mathrm{GL}_2^+(\mathbf{R})$.

(b) If we want to view this action on functions as defined by the group of linear fractional transformations, not by matrices, why should we change the definition of the action by multiplying the formula by $(ad-bc)^{k/2}$? (See Exercise 2.3.)

5. For each $N \geq 1$, the *principal congruence subgroup of level* N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

where the matrix congruence is componentwise. This is the kernel of the reduction homomorphism $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$, so $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbf{Z})$ with finite index.

Prove $\Gamma(2)$ is generated by the matrices $-I_2$, $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. (Hint: Instead of the usual division algorithm in the first proof of Theorem 3.3, use a modified division algorithm: $a = bq + r$ where $|r| \leq |b/2|$ and possibly $r < 0$.)

4. Eisenstein Series and q-expansions

The most basic example of a nonconstant modular form for $\mathrm{SL}_2(\mathbf{Z})$ is an Eisenstein series.

Definition 4.1. For even $k \geq 4$, the weight k *Eisenstein series* is

$$G_k(\tau) := \sum_{\substack{(m,n) \in \mathbf{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}.$$

Our goal is to prove G_k is a modular form of weight k for $\mathrm{SL}_2(\mathbf{Z})$. The definition of $G_k(\tau)$ makes sense for odd $k \geq 3$, but in that case the series vanishes since the terms at (m,n) and $(-m,-n)$ cancel, so it is boring. (We already saw the only modular form of odd weight for $\mathrm{SL}_2(\mathbf{Z})$ is 0.)

First we prove absolute convergence.

Lemma 4.2. *For each $\tau \in \mathfrak{h}$ there is a $\delta = \delta_\tau \in (0,1)$ such that*

$$|m\tau + n| \geq \delta |mi + n|$$

for all $m, n \in \mathbf{Z}$.

Proof. If $m = 0$ then the desired inequality holds for all n provided we use $\delta \in (0,1)$.

If $m \neq 0$, then $|m\tau + n| \geq \delta|mi + n|$ is equivalent to $|\tau + n/m| \geq \delta|i + n/m|$, which in turn is equivalent to

$$\left| \frac{\tau + n/m}{i + n/m} \right| \geq \delta.$$

Rather than working with rational n/m, let's treat this as a task in real variables: set $f_\tau \colon \mathbf{R} \to \mathbf{R}$ by $f_\tau(x) = |(\tau - x)/(i - x)|$, so $f_\tau(x) > 0$ for all x. This is a continuous function and $f_\tau(x) \to 1$ as $x \to \pm\infty$. Therefore there is a large positive number R (depending on τ) such that $f_\tau(x) \geq 1/2$ for $|x| > R$. For $x \in [-R, R]$, positivity of $f_\tau(x)$ implies by compactness of $[-R, R]$ that there is some $c > 0$ such that $f_\tau(x) \geq c_\delta$ for all $x \in [-R, R]$. Therefore $f_\tau(x) \geq \delta$ for all $x \in \mathbf{R}$ when $\delta = \min(1/2, c)$. $\square$

Theorem 4.3. *The Eisenstein series $G_k(\tau)$ is absolutely convergent: for each $\tau \in \mathfrak{h}$, the series $\sum_{(m,n) \neq (0,0)} 1/|m\tau + n|^k$ converges.*

Proof. Let $\delta = \delta_\tau$ be chosen as in Lemma 4.2. Then

$$\frac{1}{|m\tau + n|^k} \le \frac{1}{\delta^k |mi + n|^k} = \frac{1}{\delta^k \sqrt{m^2+n^2}^k}.$$

The exponent $k/2$ is greater than 1, so absolute convergence of $G_k(\tau)$ follows from absolute convergence of $\sum_{(m,n)\neq(0,0)} 1/\sqrt{m^2+n^2}^k$ for $k > 2$, which is proved in Section B as a special case of convergence of a lattice sum in any number of dimensions. □

Theorem 4.4. *For even $k \ge 4$, the Eisenstein series G_k is a modular form of weight k for $\mathrm{SL}_2(\mathbf{Z})$.*

Proof. By Theorem 4.3, $G_k(\tau)$ makes sense for each τ and the order of summation can be rearranged by absolute convergence. To prove G_k is holomorphic, we want to derive this from each term $1/(m\tau+n)^k$ in the series being holomorphic in τ. We will use a fundamental result of complex analysis about limits of holomorphic functions being holomorphic: if a sequence of holomorphic functions $\{f_n\}$ on a common domain $\Omega \subset \mathbf{C}$ converges uniformly on compact subsets of Ω then the pointwise limit $f(z) = \lim_{n\to\infty} f_n(z)$ is holomorphic on Ω.[6]

To apply this result to G_k, we will use a strengthening of Lemma 4.2: on each half-strip of the form $S_{a,b} = \{x+iy \in \mathfrak{h} : |x| \le a, y \ge b\}$ where $a > 0$ and $b > 0$, a value of δ can be chosen in Lemma 4.2 that works for all τ in $S_{a,b}$. The proof that such δ exists is left to the reader as an exercise (Exercise 4.1). Using this δ in the proof of Theorem 4.3 shows $G_k(\tau)$ converges uniformly on each $S_{a,b}$ by the Weierstrass M-test: the series $\sum_{(m,n)\neq(0,0)} 1/|m\tau+n|^k$ for $\tau \in S_{a,b}$ is bounded above termwise by $\sum_{(m,n)\neq(0,0)} 1/\delta^k\sqrt{m^2+n^2}^k$, which is independent of τ. Every compact subset of $\mathfrak{h}$ is contained in some $S_{a,b}$, so G_k converges uniformly on compact subsets of $\mathfrak{h}$ and thus is holomorphic.

To prove G_k satisfies the modularity condition with weight k, Corollary 3.6 tells us we have to check just two cases: $G_k(\tau+1) \overset{?}{=\!=} G_k(\tau)$ and $G_k(-1/\tau) \overset{?}{=} \tau^k G_k(\tau)$. For the first condition,

$$G_k(\tau+1) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m(\tau+1)+n)^k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+(m+n))^k}.$$

As (m,n) runs over $\mathbf{Z}^2 - \{(0,0)\}$, so does $(m, m+n)$, so absolute convergence of the Eisenstein series lets us rearrange the terms:

$$G_k(\tau+1) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+(m+n))^k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^k} = G_k(\tau).$$

For the second condition,

$$G_k(-1/\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(-m/\tau+n)^k} = \tau^k \sum_{(m,n)\neq(0,0)} \frac{1}{(n\tau-m)^k}.$$

This last series is $G_k(\tau)$ by rearranging terms, so $G_k(-1/\tau) = \tau^k G_k(\tau)$.

The final property we have to check is behavior of $G_k(\tau)$ as $\tau \to i\infty$. We can assume $\operatorname{Im}\tau \ge 1$, and since $G_k(\tau+1) = G_k(\tau)$ we may also assume $|\operatorname{Re}(\tau)| \le 1/2$ as $\tau \to \infty$

[6]The analogue of this in real analysis is false: the Stone–Weierstrass theorem implies $|x|$ is a uniform limit of polynomials on $(-1,1)$, and polynomials are real-analytic but $|x|$ is not real-analytic on $(-1,1)$ because there's a problem at 0.

This is the half-strip $S_{1,1}$ described earlier in the proof, so there is some $\delta > 0$ such that $|m\tau + n| \geq \delta|mi + n|$ for all $\tau \in S_{1,1}$ and $m, n \in \mathbf{Z}$.

Rearrange the terms of $G_k(\tau)$:

$$G_k(\tau) = \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0}\sum_{n\in\mathbf{Z}} \frac{1}{(m\tau+n)^k} = 2\sum_{n\geq 1}\frac{1}{n^k} + 2\sum_{m\geq 1}\sum_{n\in\mathbf{Z}}\frac{1}{(m\tau+n)^k}, \tag{4.1}$$

where we write the sum over nonzero n and outer sum over nonzero m as twice a sum over positive n and positive m using evenness of k. We will show the double series, where every term has τ in it, tends to 0 as $\tau \to \infty$, so $G_k(\tau) \to 2\sum_{n\geq 1} 1/n^k$ as $\tau \to i\infty$.

For any $N \geq 1$,

$$\begin{aligned}\sum_{m\geq 1}\sum_{n\in\mathbf{Z}}\frac{1}{|m\tau+n|^k} &= \sum_{m+|n|\leq N}\frac{1}{|m\tau+n|^k} + \sum_{m+|n|>N}\frac{1}{|m\tau+n|^k}\\ &\leq \sum_{m+|n|\leq N}\frac{1}{|m\tau+n|^k} + \frac{1}{\delta^k}\sum_{m+|n|>N}\frac{1}{|mi+n|^k}.\end{aligned}$$

Since $\sum_{m\geq 1, n\in\mathbf{Z}} 1/|m+ni|^k$ converges, for any $\varepsilon > 0$ the tail $\sum_{m+|n|>N} 1/|mi+n|^k$ is less than ε if N is sufficiently large and this doesn't involve τ. For such a choice of N, the finite series $\sum_{m+|n|\leq N} 1/|m\tau+n|^k$ is less than ε if $\operatorname{Im}\tau$ is sufficiently large. Thus the double series in (4.1) is less than 2ε if $\operatorname{Im}\tau$ is sufficiently large. □

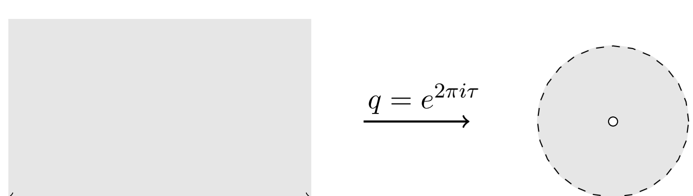
We saw in Example 1.4 that every modular form satisfies $f(\tau+1) = f(\tau)$. The function $e^{2\pi i\tau}$ also satisfies this periodicity relation, and the standard way to write down modular forms is through a power series in $e^{2\pi i\tau}$.

Theorem 4.5. *If $f\colon H \to \mathbf{C}$ is holomorphic, $f(\tau+1) = f(\tau)$ for all τ, and f is bounded as $\tau \to \infty$ then there are $a_n \in \mathbf{C}$ for $n \geq 0$ such that*

$$f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$$

for all $\tau \in \mathfrak{h}$. In particular, $f(\tau)$ has a limit as $\tau \to i\infty$.

Proof. For $\tau \in \mathfrak{h}$ set $q(\tau) = e^{2\pi i\tau}$. Writing $\tau = x + iy$, we have $q(\tau) = e^{-2\pi y}e^{2\pi i x}$, so $|q(\tau)| = e^{-2\pi y} \in (0,1)$. Thus $q(\tau)$ lies in the punctured unit disc $D' = \{q \in \mathbf{C} : 0 < |q| < 1\}$, and conversely each point in D' can be written as $e^{2\pi i\tau}$ for a discrete set of values $\tau \in \mathfrak{h}$. The mapping $\mathfrak{h} \to D'$ given by $q(\tau)$ is surjective and locally invertible: if we write $q_0 \in D'$ as $e^{2\pi i\tau_0}$ then any q sufficiently close to q_0 can be written as $e^{2\pi i\tau}$ for a unique τ near τ_0. This mapping is pictured below. Note $\tau \to i\infty$ in $\mathfrak{h}$ corresponds to $q \to 0$ in D'.



Convert the function $f\colon \mathfrak{h} \to \mathbf{C}$ into a function $\widetilde{f}\colon D' \to \mathbf{C}$ by defining $\widetilde{f}(q) = f(\tau)$ for any $\tau \in \mathfrak{h}$ that makes $e^{2\pi i\tau} = q$. This is well-defined because if $e^{2\pi i\tau'} = q$ then $\tau' = \tau + n$ for some $n \in \mathbf{Z}$, so $f(\tau') = f(\tau+n) = f(\tau)$ due to the relation $f(\tau+1) = f(\tau)$ for all $\tau \in \mathfrak{h}$. Since f is holomorphic, we can prove $\widetilde{f}$ is holomorphic by computing the derivative of $\widetilde{f}$:

for each $q_0 \in D'$, write $q_0 = e^{2\pi i\tau_0}$. Then any q near q_0 is $e^{2\pi i\tau}$ for a unique τ near τ_0, and $q \to q_0$ is equivalent to $\tau \to \tau_0$. Thus

$$\frac{\widetilde{f}(q) - \widetilde{f}(q_0)}{q - q_0} = \frac{f(\tau) - f(\tau_0)}{q - q_0} = \frac{f(\tau) - f(\tau_0)}{\tau - \tau_0}\frac{\tau - \tau_0}{e^{2\pi i\tau} - e^{2\pi i\tau_0}}.$$

As $\tau \to \tau_0$, the right side tends to $f'(\tau_0)/(2\pi i e^{2\pi i\tau_0}) = f'(\tau_0)/(2\pi i q_0)$. (This formula for $f'(q_0)$ is intuitive by the chain rule: $d\widetilde{f}/dq = (df/d\tau)(d\tau/dq) = f'(\tau)(d\tau/d(e^{2\pi i\tau})) = f'(\tau)/(2\pi i q)$.)

The boundedness of $f(\tau)$ as $\tau \to i\infty$ implies boundedness of $\widetilde{f}(q)$ as $q \to 0$. An important theorem in complex analysis, Riemann's removable singularities theorem, says a holomorphic function on a punctured neighborhood $\{z : 0 < |z - a| < r\}$ of a point a that is bounded on a small neighborhood of a (i) has a limit as $z \to a$ and (ii) the extended function set equal to the limit at $z = a$ is holomorphic at a. Therefore the boundedness of $\widetilde{f}(q)$ as $q \to 0$ implies $\widetilde{f}$ is holomorphic at 0. Thus $\widetilde{f}$ has a power series expansion at 0, say $\sum_{n\geq 0} a_n q^n$. Since $\widetilde{f}$ is holomorphic on the whole open unit disc $D = \{q \in \mathbf{C} : |q| < 1\}$, another basic theorem from complex analysis guarantees that $\sum_{n\geq 0} a_n q^n$ converges on all of D: a holomorphic function on an open disc has its series at the center converge on the whole disc. Therefore

$$f(\tau) = \widetilde{f}(e^{2\pi i\tau}) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$$

for all $\tau \in \mathfrak{h}$. □

Definition 4.6. The *q-expansion* of a modular form $f(\tau)$ is the series $\sum_{n\geq 0} a_n q^n$ for which $f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$. The coefficients a_n in the q-expansion are called the *Fourier coefficients* of f.

A q-expansion is not merely a formal object: the equation $f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i n\tau}$ is analytic on both sides, with the right side convergent for every $\tau \in \mathfrak{h}$. When writing a modular form $f(\tau)$ as its q-expansion, it is a common abuse of notation to write the function as $f(q)$, using the same letter f with the new variable $q = e^{2\pi i\tau}$.

The constant term a_0 in the q-expansion is $f(i\infty)$ when f is a function of τ and $f(0)$ when f is a function of q. While the q-expansion of f encodes the relation $f(\tau + 1) = f(\tau)$, the other relation $f(-1/\tau) = \tau^k f(\tau)$ is not visible in a q-expansion. If we are given a new power series converging on the open unit disc, there is usually no simple way to show if it is the q-expansion of a modular form without further information. The definition of a modular form is awkward to formulate directly in terms of q-expansions.

For the rest of this section we will work out the q-expansion of the Eisenstein series G_k. We already saw in the proof of Theorem 4.4 that the constant term of the q-expansion is $2\sum_{n\geq 1} 1/n^k$. For every complex number s with $\mathrm{Re}(s) > 1$, the *Riemann zeta-function* at s is $\zeta(s) := \sum_{n\geq 1} 1/n^s$. This series is absolutely and uniformly convergent on compact subsets of $\{s : \mathrm{Re}(s) > 1\}$, so $\zeta(s)$ is holomorphic on $\{s : \mathrm{Re}(s) > 1\}$. The constant term of $G_k(\tau)$ is $2\zeta(k)$, and long before Riemann worked with $\zeta(s)$ Euler showed $\zeta(k)$ is a rational multiple of π^k when k is a positive even integer, *e.g.*, $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

Theorem 4.7. *For even $k \geq 4$, the q-expansion of $G_k(\tau)$ is* $2\zeta(k) + \dfrac{2(2\pi i)^k}{(k-1)!}\displaystyle\sum_{n\geq 1} \sigma_{k-1}(n) q^n$, *where* $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Proof. We recall (4.1):

$$G_k(\tau) = 2\sum_{n\geq 1}\frac{1}{n^k} + 2\sum_{m\geq 1}\left(\sum_{n\in\mathbf{Z}}\frac{1}{(m\tau+n)^k}\right) = 2\zeta(k) + 2\sum_{m\geq 1}\left(\sum_{n\in\mathbf{Z}}\frac{1}{(m\tau+n)^k}\right).$$

The inner sum has the form

$$\sum_{n\in\mathbf{Z}}\frac{1}{(w+n)^k} \tag{4.2}$$

where $w = m\tau \in \mathfrak{h}$. This series, by its very shape, is a periodic function of w: its values at w and $w+1$ are equal, so we might think it could be written in terms of $e^{2\pi i w}$. We will prove

$$\sum_{n\in\mathbf{Z}}\frac{1}{(w+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1} n^{k-1}e^{2\pi i n w}. \tag{4.3}$$

for all $w \in \mathfrak{h}$ and plugging this into the formula for $G_k(\tau)$ using $w = m\tau$ as m varies will produce the q-expansion of $G_k(\tau)$.

To analyze a series like (4.2) we will use a beautiful result in Fourier analysis that expresses the sum of one function over $\mathbf{Z}$ as the sum of another function over $\mathbf{Z}$: the Poisson summation formula. This summation formula says that if $f\colon \mathbf{R}\to\mathbf{C}$ is a suitably nice function then

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}}\widehat{f}(n),$$

where $\widehat{f}\colon \mathbf{R}\to\mathbf{C}$ is the *Fourier transform* of f:

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{2\pi i x y}\,dx.$$

What does "suitably nice" mean?

First we need the function to have a Fourier transform. If $f\colon \mathbf{R}\to\mathbf{C}$ is absolutely integrable on $\mathbf{R}$, meaning $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$, then the Fourier transform of f is defined since $|e^{2\pi i x y}| = 1$. For the function $1/(w+x)^k$ with $w\in\mathfrak{h}$, writing $w = a+bi$, we have

$$\frac{1}{|w+x|^k} = \frac{1}{|(a+x)+bi|^k} = \frac{1}{((a+x)^2+b^2)^{k/2}},$$

so $1/(w+x)^k$ is absolutely integrable for $k\geq 2$. Thus the Fourier transform of $1/(w+x)^k$ makes sense for all $y\in\mathbf{R}$.

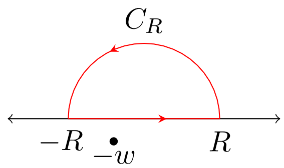
The Poisson summation formula is valid for any function $f\colon \mathbf{R}\to\mathbf{C}$ for which f and its Fourier transform $\widehat{f}$ are both continuous and absolutely integrable on $\mathbf{R}$. Clearly $1/(w+x)^k$ is continuous, and we showed it is absolutely integrable. It remains to compute its Fourier transform and check it is continuous and absolutely integrable.

Letting $\varphi_w(x) = 1/(w+x)^k$,

$$\widehat{\varphi_w}(y) = \int_{\mathbf{R}}\frac{e^{2\pi i x y}}{(w+x)^k}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{2\pi i x y}}{(w+x)^k}\,dx. \tag{4.4}$$

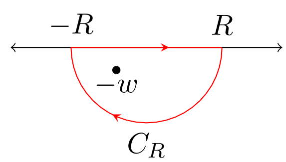
We will calculate this integral using the residue theorem from complex analysis. Complexify the integrand to $h(z) = e^{2\pi i z y}/(w+z)^k$ for $z\in\mathbf{C}$. This has a kth order pole at $-w$, which is a point in the lower half-plane since $w\in\mathfrak{h}$.

The numerator $e^{2\pi i z y}$ in $h(z)$ has absolute value $e^{-2\pi \operatorname{Im}(z) y}$, so if $y \geq 0$ we want to integrate $h(z)$ along $[-R, R]$ and then counterclockwise along the semicircle in the *upper* half plane with R and $-R$ as endpoints (figure below on the left), since in the upper half-plane $|e^{-2\pi \operatorname{Im}(z)y}| \leq 1$. If $y < 0$, we want to integrate $h(z)$ along $[-R, R]$ and then clockwise along the semicircle in the *lower* half-plane connecting R to $-R$ (figure below on the right), since $|e^{-2\pi \operatorname{Im}(z)y}| \leq 1$ on this semicircle. Let C_R in each case be the indicated contour of integration.



Contour for $y \geq 0$

Contour for $y < 0$



By the residue theorem, for $y \geq 0$

$$\int_{C_R} h(z)\, dz = 0$$

for all $R > 0$, and it is left to the reader to check the integral of $h(z)$ along the semicircular part of C_R tends to 0 as $R \to \infty$, so

$$\int_{-R}^{R} h(x)\, dx \to 0$$

as $R \to \infty$. This says $\widehat{\varphi_w}(y) = 0$ if $y \geq 0$. For $y < 0$, using the second contour as C_R,

$$\int_{C_R} h(z)\, dz = -2\pi i \operatorname{Res}_{z=-w} h(z)$$

by the residue theorem if R is large enough that the pole of $h(z)$ is inside C_R. There is a minus sign in front of the residue because we are integrating clockwise instead of counterclockwise in order to be integrating in the natural direction along the real axis. Check the integral along the semicircle in C_R tends to 0 as $R \to \infty$, so

$$\int_{-R}^{R} h(x)\, dx \to -2\pi i \operatorname{Res}_{z=-w} h(z) = -2\pi i \operatorname{Res}_{z=-w} \frac{e^{2\pi i z y}}{(w+z)^k} = -2\pi i e^{-2\pi i w y} \operatorname{Res}_{z=0} \frac{e^{2\pi i z y}}{z^k}.$$

For any $a \in \mathbf{C}$, $\operatorname{Res}_{z=0}(e^{az}/z^k) = a^{k-1}/(k-1)!$, so $\widehat{\varphi_w}(y) = -2\pi i e^{-2\pi i w y} \operatorname{Res}_{z=0} e^{2\pi i z y}/z^k = -2\pi i e^{-2\pi i w y}(2\pi i y)^{k-1}/(k-1)!$.

Our calculation of (4.4) can be summarized as

$$\widehat{\varphi_w}(y) = \begin{cases} 0, & \text{if } y \geq 0, \\ \frac{-(2\pi i)^k}{(k-1)!} y^{k-1} e^{-2\pi i w y}, & \text{if } y < 0. \end{cases} \tag{4.5}$$

As a function of y, (4.5) is continuous[7] and, up to a constant, the integral of $|\widehat{\varphi_w}(y)|$ over $\mathbf{R}$ is bounded above by $\int_{-\infty}^{0} |y|^{k-1} e^{2\pi (\operatorname{Im} w) y}\, dy$, which is finite.

[7]There is a general theorem in Fourier analysis that a function that is absolutely integrable has a Fourier transform that is continuous, so the continuity of $\widehat{\varphi_w}$ was predictable.

It is therefore legal to apply Poisson summation to the function φ_w:

$$\sum_{n \in \mathbf{Z}} \frac{1}{(w+n)^k} = \sum_{n \in \mathbf{Z}} \widehat{\varphi_w}(n) = \sum_{n \leq -1} \widehat{\varphi_w}(n) = -\frac{(2\pi i)^k}{(k-1)!} \sum_{n \leq -1} n^{k-1} e^{-2\pi i w n}.$$

Replacing n with $-n$ for $n \geq 1$,

$$\sum_{n \in \mathbf{Z}} \frac{1}{(w+n)^k} = -\frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} (-n)^{k-1} e^{2\pi i w n} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i w n}.$$

which is (4.3) except we have $(-2\pi i)^k$ instead of $(2\pi i)^k$. The factor $(-2\pi i)^k$, for $k \geq 2$ even or odd, is the right one. In our application k is even, so the sign doesn't matter.

Returning now to the equation at the start of the proof,

$$\begin{aligned} G_k(\tau) &= 2\zeta(k) + 2\sum_{m \geq 1} \left(\sum_{n \in \mathbf{Z}} \frac{1}{(m\tau + n)^k} \right) \\ &= 2\zeta(k) + 2\sum_{m \geq 1} \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i \tau (mn)} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m \geq 1} \sum_{n \geq 1} n^{k-1} q^{mn}. \end{aligned}$$

Writing mn as N, summing over positive integers m and n is the same as summing over positive integers m and N *with the constraint* that $m \mid N$, so

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{N \geq 1} \left(\sum_{n \mid N} n^{k-1} \right) q^N = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{N \geq 1} \sigma_{k-1}(N) q^N.$$

□

Remark 4.8. In most treatments of modular forms, the q-expansion of $G_k(\tau)$ is derived not using Poisson summation, but using a more elementary method involving the partial fraction decomposition of $\pi \cot(\pi z)$. We use the technique of Poisson summation since it's good to get familiar with it. We'll use Poisson summation later to construct a special modular form of weight 12.

Euler's formula for $\zeta(k)$ when $k \geq 2$ is even is

$$\zeta(k) = \frac{(2\pi)^k (-1)^{k/2+1}}{k!} \frac{B_k}{2} = -\frac{(2\pi i)^k}{(k-1)!} \frac{B_k}{2k}, \tag{4.6}$$

where B_k is the kth Bernoulli number: it is a rational number appearing in the power series

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} \frac{B_k}{k!} x^k = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$$

The table below lists the first few Bernoulli numbers. The early data suggest $B_k = 0$ for odd $k > 1$, which is true. The early data also suggest $|B_k|$ is small, but actually $|B_k| \to \infty$ as $k \to \infty$.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$	0	$\frac{7}{6}$

By Theorem 4.7 and (4.6),

$$G_k(\tau) = 2\zeta(k) - \frac{4k\zeta(k)}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

For arithmetic applications it is convenient to scale $G_k(\tau)$ so that its constant term is 1.

Definition 4.9. For even $k \geq 4$, define the *normalized Eisenstein series of weight* k to be

$$E_k(\tau) = E_k(q) := \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \tag{4.7}$$

Using the table of values of Bernoulli numbers, some special cases of (4.7) are

$$\begin{aligned}
E_4(\tau) &= 1 + 240q + 2160q^2 + 6720q^3 + \dots \\
E_6(\tau) &= 1 - 504q - 16632q^2 - 122976q^3 - \dots \\
E_8(\tau) &= 1 + 480q + 61920q^2 + 1050240q^3 + \dots \\
E_{10}(\tau) &= 1 - 264q - 135432q^2 - 5196576q^3 - \dots \\
E_{12}(\tau) &= 1 + \frac{65520}{691}q + \frac{134250480}{691}q^2 + \frac{11606736960}{691}q^3 + \dots \\
E_{14}(\tau) &= 1 - 24q - 196632q^2 - 38263776q^3 - \dots
\end{aligned}$$

Since $2k/B_k \in \mathbf{Z}$ for $k = 4, 6, 8, 10$, and 14, all Fourier coefficients of $E_k(\tau)$ are integers for these k.

The product of modular forms of weight k and ℓ is easily seen to be a modular form of weight $k+\ell$, and we can find the q-expansion of the product by multiplying the q-expansions of the two modular forms. For example,

$$\begin{aligned}
E_4(\tau)^2 &= 1 + 480q + 61920q^2 + 1050240q^3 + \dots \text{ has weight 8}, \\
E_4(\tau)E_6(\tau) &= 1 - 264q - 135432q^2 - 5196576q^3 + \dots \text{ has weight 10}, \\
E_4(\tau)^3 &= 1 + 720q + 179280q^2 + 16954560q^3 + \dots \text{ has weight 12}, \\
E_6(\tau)^2 &= 1 - 1008q + 220752q^2 + 16519104q^3 + \dots \text{ has weight 12}.
\end{aligned}$$

From the initial parts of q-expansions, it looks like $E_8 = E_4^2$ and $E_{10} = E_4E_6$. In weight 12, the modular forms E_{12}, E_4^3, and E_6^2 are all different and are not scalar multiples of each other since their constant terms all equal 1.

The explanation for identities like $E_8 = E_4^2$ and $E_{10} = E_4E_6$ will come from the fact that the modular forms of a fixed weight are a complex vector space that is *finite-dimensional*, whose proof is the main goal of Section 5.

While the original definition of $G_k(\tau)$ for even $k \geq 4$ makes no sense when $k = 2$, the q-expansion of $G_k(\tau)$ in Theorem 4.7 does make sense at $k = 2$!

Definition 4.10. For $\tau \in \mathfrak{h}$, define

$$G_2(\tau) = 2\zeta(2) + \frac{2(2\pi i)^2}{(2-1)!}\sum_{n\geq 1}\sigma_1(n)q^n = \frac{\pi^2}{3} - 8\pi^2\sum_{n\geq 1}\sigma_1(n)q^n$$

and $E_2(\tau) = \dfrac{G_2(\tau)}{2\zeta(2)} = 1 - 24\displaystyle\sum_{n\geq 1}\sigma_1(n)q^n$, where $q = e^{2\pi i\tau}$.

The series $G_2(\tau)$ converges for all q in the open unit disc on account of the weak bound $\sigma_1(n) = \sum_{d|n} d \leq \sum_{k=1}^{n} k \sim n^2/2$. It is holomorphic in q (as all convergent power series are in a disc of convergence) and thus is also holomorphic in τ by composition. Trivially $G_2(\tau+1) = G_2(\tau)$ and $G_2(\tau) \to \pi^2/3$ as $\tau \to i\infty$. Could $G_2(-1/\tau) = \tau^2 G_2(\tau)$ for all τ, making $G_2(\tau)$ a modular form of weight 2 for $\mathrm{SL}_2(\mathbf{Z})$? No.

Theorem 4.11. *For all $\tau \in \mathfrak{h}$, $G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i\tau$. Equivalently, $E_2(-1/\tau) = \tau^2 G_2(\tau) - (6i/\pi)\tau$.*

Proof. This is a project. □

We will see in Section 5 that the only modular form of weight 2 for $\mathrm{SL}_2(\mathbf{Z})$ is 0.

Exercises.

1. In the proof of Lemma 4.2, for each half-strip $S_{a,b} = \{x+iy : |x| \leq a, y \geq b\}$ in $\mathfrak{h}$, where $a > 0$ and $b > 0$, show there is a $\delta > 0$ such that $|m\tau + n| \geq \delta|mi + n|$ for all $\tau \in S_{a,b}$ and all $m, n \in \mathbf{Z}$. That is, δ in Lemma 4.2 can be chosen uniformly in $S_{a,b}$.
2. For even $k \geq 4$, show
$$G_k(\tau) = \zeta(k) \sum_{(m,n)=1} \frac{1}{(m\tau+n)^k},$$
so $E_k(\tau) = (1/2)\sum_{(m,n)=1}(m\tau+n)^{-k}$.
3. Let M be a positive integer and $k \geq 4$ an even integer. Show
$$\sum_{\substack{(m,n)\in M\mathbf{Z}\times\mathbf{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)e^{2\pi i M n\tau}.$$

5. Dimensions of Spaces of Modular Forms

Let M_k denote the set of all weight k modular forms for $\mathrm{SL}_2(\mathbf{Z})$. It is a vector space over $\mathbf{C}$. In this section, we show each M_k is finite-dimensional and write down an explicit dimension formula.

The proof will fall into four parts:

(1) Prove $M_k = \{0\}$ for $k < 0$,
(2) Construct a modular form $\Delta(\tau)$ of weight 12 that is nonvanishing on $\mathfrak{h}$.[8]
(3) Use (1) and (2) to compute $\dim M_k$ for $0 \leq k \leq 10$.
(4) Use (2) and (3) to compute $\dim M_k$ for $k \geq 12$.

Theorem 5.1. *If $k < 0$ then $M_k = \{0\}$.*

Proof. Pick $f \in M_k$ and write its q-expansion as $\sum_{n\geq 0} a_n q^n$. We will prove each Fourier coefficient a_n is 0, so $f = 0$.

The modularity condition for f and the imaginary part formula (2.2) raised to the $k/2$-power say
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \left(\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right)^{k/2} = \frac{(\mathrm{Im}\,\tau)^{k/2}}{|c\tau+d|^k},$$

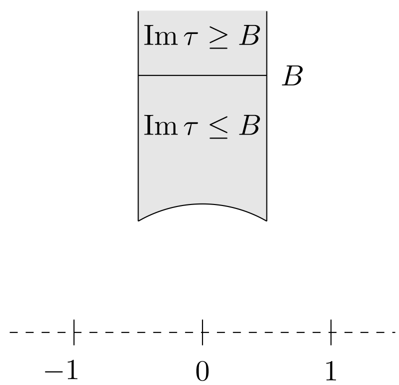
[8] The minimal positive weight for a modular form that vanishes nowhere on $\mathfrak{h}$ is 12 because if $f \in M_k$ and $k \not\equiv 0 \bmod 3$ then $f(e^{2\pi i/3}) = 0$, and if $k \not\equiv 0 \bmod 4$ then $f(i) = 0$. See Exercise 3.3 for special cases.

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. Therefore if we take the absolute value of f and multiply,

$$\left|f\left(\frac{a\tau+b}{c\tau+d}\right)\right|\left(\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right)^{k/2} = |c\tau+d|^k|f(\tau)|\frac{(\operatorname{Im}\tau)^{k/2}}{|c\tau+d|^k} = |f(\tau)|(\operatorname{Im}\tau)^{k/2}.$$

This says the continuous real-valued function $|f(\tau)|(\operatorname{Im}\tau)^{k/2}$ on $\mathfrak{h}$ is $\mathrm{SL}_2(\mathbf{Z})$-invariant. (So far we have not used $k<0$.)

Any $\mathrm{SL}_2(\mathbf{Z})$-invariant function on $\mathfrak{h}$ has all of its values achieved on the fundamental domain $\mathcal{F}$ from Section 3. Break up $\mathcal{F}$ into two parts: that with $\operatorname{Im}\tau \le B$ and that with $\operatorname{Im}\tau \ge B$ for B to be determined. See the picture below.



As $\tau \to i\infty$ in $\mathcal{F}$, $|f(\tau)|$ is bounded and $(\operatorname{Im}\tau)^{k/2} \to 0$ *because* $k<0$. Therefore $|f(\tau)|(\operatorname{Im}\tau)^{k/2} \to 0$ as $\tau \to i\infty$, so there is some $B>0$ such that $|f(\tau)|(\operatorname{Im}\tau)^{k/2} \le 1$ for $\operatorname{Im}\tau \ge B$. On $\{\tau \in \mathcal{F} : \operatorname{Im}\tau \le B\}$ the function $|f(\tau)|(\operatorname{Im}\tau)^{k/2}$ is bounded above since a continuous real-valued function on a compact set is bounded. Putting these two parts together, there is some $C>0$ such that

$$|f(x+iy)|y^{k/2} \le C \tag{5.1}$$

for all $x+iy \in \mathcal{F}$ and thus also for all $x+iy \in \mathfrak{h}$ by $\mathrm{SL}_2(\mathbf{Z})$-invariance.

Pick $y>0$. In the q-expansion $f(x+iy) = \sum_{n\ge 0} a_n q^n = \sum_{n\ge 0} a_n e^{-2\pi n y} e^{2\pi i n x}$, multiply both sides by $e^{-2\pi i m x}$ and integrate from 0 to 1:

$$\int_0^1 f(x+iy)e^{-2\pi i m x}\,dx = \sum_{n\ge 0} a_n e^{-2\pi n y}\int_0^1 e^{2\pi i n x}e^{-2\pi i m x}\,dx.$$

(Why can the series be integrated termwise?) Since $\int_0^1 e^{2\pi i n x}e^{-2\pi i m x}\,dx = \int_0^1 e^{2\pi i(n-m)x}\,dx$ is 0 for $n \ne m$ and is 1 for $n=m$, the integral produces the mth Fourier coefficient:

$$\int_0^1 f(x+iy)e^{-2\pi i m x}\,dx = a_m e^{-2\pi m y},$$

so

$$a_m = e^{2\pi m y}\int_0^1 f(x+iy)e^{-2\pi i m x}\,dx \overset{(5.1)}{\Longrightarrow} |a_m| \le e^{2\pi m y}\int_0^1 Cy^{-k/2}\,dx = \frac{Ce^{2\pi m y}}{y^{k/2}}.$$

This holds for all $y>0$. Letting $y \to 0^+$, the factor $e^{2\pi m y}$ tends to 1 and the factor $y^{k/2}$ tends to ∞ since $k<0$. Therefore $|a_m|=0$, so $a_m=0$ for all m. Thus $f=0$. □

Theorem 5.2. *There is a modular form $\Delta(\tau) \in M_{12}$ that is nonvanishing on $\mathfrak{h}$ and it has a simple zero at $i\infty$: its q-expansion starts out as $q + b_2q^2 + \cdots$.*

Using Eisenstein series it is easy to construct a modular form of weight 12 whose q-expansion starts out with q: since $E_4^3 = 1 + 720q + \cdots$ and $E_6^2 = 1 - 1008q + \cdots$, the difference $(E_4^3 - E_6^2)/1728$ has first term q in its q-expansion. What is not easy to see is that this modular form vanishes nowhere on $\mathfrak{h}$. The way we will prove Theorem 5.2 is by building a modular form of weight 12 in a different way. The argument is rather technical (it will use a twisted version of Poisson summation), so for now we will accept Theorem 5.2 as proved and see how to use it to compute the dimensions (and bases) of every M_k for $k \geq 0$. At the end of this section we will return to prove Theorem 5.2.

Theorem 5.3. *For $k = 0, 2, 4, 6, 8, 10$, $\dim M_k$ is given in the following table.*

k	0	2	4	6	8	10
$\dim M_k$	1	0	1	1	1	1

Proof. First we treat the cases $k = 4, 6, 8, 10$. Let $f \in M_k$ and $a_0 = f(i\infty)$. The difference $f(\tau) - a_0 E_k(\tau)$ lies in M_k and its q-expansion has constant term $a_0 - a_0 = 0$.

The ratio $(f - a_0 E_k)/\Delta$ lies in M_{k-12}: it is holomorphic on $\mathfrak{h}$ since $\Delta(\tau) \neq 0$ for all $\tau \in \mathfrak{h}$, it easily satisfies the modularity condition for weight $k - 12$. and as $q \to 0$ the ratio has a finite limit since $f - a_0 E_k$ has a zero at $q = 0$ and Δ has a simple zero at $q = 0$. By Theorem 5.1, $M_{k-12} = \{0\}$ since $k - 12 < 0$, so $f - a_0 E_k = 0$. Thus $f = a_0 E_k$, so $M_k = \mathbf{C}E_k$ is one-dimensional.

If $k = 0$, the constant function 1 lies in M_0 and reasoning as above with 1 in place of E_k shows $f = a_0 \cdot 1 = a_0$, so $M_0 = \mathbf{C}$.

Finally, we will prove $M_2 = 0$. Let $f \in M_2$, so $f(-1/\tau) = \tau^2 f(\tau)$ for all $\tau \in \mathfrak{h}$. Setting $\tau = i$ we get $f(i) = -f(i)$, so $f(i) = 0$. The square f^2 lies in M_4, and we already proved $M_4 = \mathbf{C}E_4$, so $f(\tau)^2 = cE_4(\tau)$ for some $c \in \mathbf{C}$ and all τ. Setting $\tau = i$ on both sides and using the q-expansion of E_4,

$$0 = cE_4(i) = c\left(1 + 240\sum_{n\geq 1} \sigma_3(n) e^{-2\pi n}\right).$$

The sum on the right is positive, so $c = 0$ and thus $f = 0$. □

Theorem 5.4. *Every space M_k is finite-dimensional. For even $k \geq 0$,*

$$\dim M_k = \begin{cases} [k/12] + 1, & \text{if } k \not\equiv 2 \bmod 12, \\ [k/12], & \text{if } k \equiv 2 \bmod 12. \end{cases}$$

Proof. We have verified the theorem for $k = 0, 2, 4, 6, 8$, and 10.

For even $k \geq 12$ and $f \in M_k$ with constant term a_0, $(f - a_0 E_k)/\Delta$ lies in M_{k-12} by the reasoning used in the proof of Theorem 5.3. Therefore $f = a_0 E_k + \Delta g$ where $g \in M_{k-12}$, so the $\mathbf{C}$-linear map $\mathbf{C} \oplus M_{k-12} \to M_k$ given by $(c, g) \mapsto cE_k + \Delta g$ is surjective. To show it is injective we show the kernel is 0: if $cE_k + \Delta g = 0$ in M_k then looking at the constant term of the q-expansion on the left implies $c = 0$, so $\Delta g = 0$, and thus $g = 0$.

Since $\mathbf{C} \oplus M_{k-12} \cong M_k$ as $\mathbf{C}$-vector spaces for $k \geq 12$, M_k is finite-dimensional with dimension $1 + \dim M_{k-12}$. The dimension formula in the theorem satisfies the same recursion, so we are done by induction on k. □

Here is an initial list of the dimensions of M_k for even $k \geq 0$. Note in particular that $\dim M_k = 1$ exactly for $k = 0, 4, 6, 8, 10$, and 14.

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26
$\dim M_k$	1	0	1	1	1	1	2	1	2	2	2	2	3	2

Example 5.5. The equations $E_8 = E_4^2$ and $E_{10} = E_4E_6$ follow from M_8 and M_{10} being one-dimensional; just check the constant terms on both sides agree.

Example 5.6. The space M_{12} has dimension 2, so E_4^3 and E_6^2 must be a basis since they are nonzero and are not scalar multiples (look at the q-expansions).

Since $E_{12} \in M_{12}$, there are complex numbers a and b such that $E_{12} = aE_4^3 + bE_6^2$. We can find a and b by looking at the constant and linear Fourier coefficients on both sides as the first and second components of a vector equation:

$$\begin{pmatrix} 1 \\ 655020/691 \end{pmatrix} = a\begin{pmatrix} 1 \\ 720 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1008 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 720 & -1008 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Using linear algebra, $a = 441/691$ and $b = 250/691$. For example, if we look at the coefficients of q^2 in E_{12}, E_4^3, and E_6^2 then

$$\frac{134250480}{691} = 179280a + 220752b.$$

Since modular forms lie in finite-dimensional spaces but their q-expansions have infinitely many Fourier coefficients, there is some redundancy in the coefficients: knowing a suitable finite list of Fourier coefficients is enough to determine the modular form. The following theorem is one version of this idea.

Theorem 5.7. *For each even $k \geq 0$ there is an $R \geq 0$ such that the first R Fourier coefficients of any weight k modular form for $\mathrm{SL}_2(\mathbf{Z})$ determine the form.*

Proof. Let $L_j \colon M_k \to \mathbf{C}^j$ by sending each modular form to the vector of its first j Fourier coefficients:

$$L_j(f) = (a_0, \ldots, a_{j-1}).$$

The kernels $\ker(L_j)$ are a decreasing sequence of subspaces of M_k: $\ker(L_{j+1}) \subset \ker(L_j)$. Since M_k is finite-dimensional, the kernel subspaces must eventually stabilize, say $\ker(L_R) = \ker(L_j)$ for all $j \geq R$.

This implies $\ker(L_R) = 0$, by contradiction. If this kernel were nonzero, there is a nonzero $f \in M_k$ whose first R coefficients vanish. Some later coefficient is nonzero, say the R'-th coefficient, so $\ker(L_{R'})$ is a proper subspace of $\ker(L_R)$, which contradicts the stabilization. Thus $\ker(L_R) = 0$, so L_R is injective and that means each $f \in M_k$ is determined by its first R Fourier coefficients. □

Clearly R has to be at least as large as the dimension of M_k. It turns out that this minimal choice always works, but that is not obvious and we omit the proof.

Now we return to the proof of Theorem 5.2, which says there is a weight 12 modular form that is nonvanishing on $\mathfrak{h}$ with a simple zero at $i\infty$. The construction of this modular form will use a "twisted" version of Poisson summation. The usual Poisson summation formula says

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)$$

for suitably nice functions $f \colon \mathbf{R} \to \mathbf{C}$ (*e.g.*, it suffices for both f and $\widehat{f}$ to be continuous and absolutely integrable). A twisted version of Poisson summation is the following equality of sums over odd integers:

$$\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}f(n)=\frac{i}{2}\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}\widehat{f}(n/4). \tag{5.2}$$

This formula can be proved using the ordinary Poisson summation formula on suitable auxiliary functions (Exercise 5.5c). We will use (5.2) for the function $f(x)=xe^{-\pi ax^2}$, where $a>0$. Its Fourier transform is $\widehat{f}(y)=(-iy/a^{3/2})e^{-\pi y^2/a}$ (Exercise 5.5b). Both $f(x)$ and $\widehat{f}(y)$ are continuous and absolutely integrable on $\mathbf{R}$, which suffices to justify using (5.2). Thus

$$\begin{aligned}\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi an^2} &= \frac{i}{2}\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}\frac{-i(n/4)}{a^{3/2}}e^{-\pi n^2/16a}\\ &= \frac{1}{8a^{3/2}}\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi n^2/16a}.\end{aligned}$$

Replacing a with $a/4$ throughout,

$$\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi an^2/4}=\frac{1}{a^{3/2}}\sum_{\substack{n\in\mathbf{Z}\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi n^2/4a}.$$

In each sum, the terms at n and $-n$ are equal, so combine the terms and divide by 2:

$$\sum_{\substack{n\geq 1\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi an^2/4}=\frac{1}{a^{3/2}}\sum_{\substack{n\geq 1\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi n^2/4a}. \tag{5.3}$$

For $\tau\in\mathfrak{h}$, define

$$\theta(\tau)=\sum_{\substack{n\geq 1\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{\pi in^2\tau/4}=e^{\pi i\tau/4}-3e^{\pi i9\tau/4}+5e^{\pi i25\tau/4}-\cdots.$$

Writing $\tau=x+iy$,

$$\theta(x+iy)=\sum_{\substack{n\geq 1\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi n^2y/4}e^{\pi in^2x/4},$$

which converges very rapidly; it is holomorphic on $\mathfrak{h}$ (as the series converges uniformly on compact subsets of $\mathfrak{h}$) and $\theta(i\infty)=0$. Along the imaginary axis

$$\begin{aligned}\theta(iy) &= \sum_{\substack{n\geq 1\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi n^2y/4}\\ &= \frac{1}{y^{3/2}}\sum_{\substack{n\geq 1\\ n\text{ odd}}}(-1)^{(n-1)/2}ne^{-\pi n^2/4y}\text{ by (5.3)}\\ &= \frac{1}{y^{3/2}}\theta(i/y)\\ &= \frac{1}{y^{3/2}}\theta\left(-\frac{1}{iy}\right).\end{aligned}$$

Therefore $\theta(-1/iy) = y^{3/2}\theta(iy)$. Raise both sides to the 8th power:

$$\theta\left(\frac{-1}{iy}\right)^8 = y^{12}\theta(iy)^8 = (iy)^{12}\theta(iy)^8.$$

It follows from this that $\theta(-1/\tau)^8 = \tau^{12}\theta(\tau)^8$ on $\mathfrak{h}$ since both sides are holomorphic and we proved they are equal on the imaginary axis in $\mathfrak{h}$, so they must be equal everywhere.

It is left to the reader to check $\theta(\tau+1) = \frac{1+i}{\sqrt{2}}\theta(\tau)$ (Exercise 5.6). Since $(1+i)/\sqrt{2}$ is an 8th root of unity, $\theta(\tau+1)^8 = \theta(\tau)^8$.

Definition 5.8. For $\tau \in \mathfrak{h}$, define $\Delta(\tau) = \theta(\tau)^8$.

The function $\Delta(\tau)$ is holomorphic on $\mathfrak{h}$, since $\theta(\tau)$ is, and we proved $\Delta(\tau+1) = \Delta(\tau)$ and $\Delta(-1/\tau) = \tau^{12}\Delta(\tau)$. Therefore $\Delta \in M_{12}$, and since $\theta(i\infty) = 0$ also $\Delta(i\infty) = 0$.

Proof. (of Theorem 5.2) To show the q-expansion of Δ starts with q, since $\Delta(i\infty) = 0$ we know there is a q-expansion $\Delta(\tau) = \sum_{n\geq 1} a_n q^n = \sum_{n\geq 1} a_n e^{2\pi i n\tau}$ and we want to show $a_1 = 1$ (most importantly, that $a_1 \neq 0$). Since $\theta(\tau)$ is defined as a power series in $e^{\pi i\tau/4}$ whose first term is $e^{\pi i\tau/4}$, its 8th power has first term $(e^{\pi i\tau/4})^8 = e^{2\pi i\tau} = q$.

To prove Δ is nonvanishing on $\mathfrak{h}$, we will prove θ is novanishing on $\mathfrak{h}$. Suppose $\theta(\tau_0) = 0$ for some τ_0. Then $\theta(\gamma\tau_0) = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, so we may assume $\tau_0 \in \mathcal{F}$ (the fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$). In the equation

$$0 = \theta(\tau_0) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} (-1)^{(n-1)/2} n e^{\pi i n^2\tau_0/4}$$

bring the term at $n = 1$ over to the left side and take absolute values:

$$|e^{\pi i\tau_0/4}| \leq \sum_{\substack{n\geq 3\\ n \text{ odd}}} n|e^{\pi i n^2\tau_0/4}| \tag{5.4}$$

Set $\tau_0 = x_0 + iy_0$, so $y_0 \geq \sqrt{3}/2$ because $\tau_0 \in \mathcal{F}$. Then (5.4) becomes

$$e^{-\pi y_0/4} \leq \sum_{\substack{n\geq 3\\ n \text{ odd}}} n e^{-\pi n^2 y_0/4},$$

so

$$1 \leq \sum_{\substack{n\geq 3\\ n \text{ odd}}} n e^{-\pi(n^2-1)y_0/4} \leq \sum_{\substack{n\geq 3\\ n \text{ odd}}} n e^{-\pi(n^2-1)\sqrt{3}/8}.$$

The sum on the right is rapidly convergent and without caring about error estimates the sum of the first few terms is approximately .013, which is much less than 1, so it appears we have a contradiction.

To make that last step rigorous, we will prove $\sum_{\text{odd } n\geq 3} n e^{-\pi(n^2-1)\sqrt{3}/8} < 1$. Writing odd $n \geq 3$ as $2m+1$ for $m \geq 1$ and doing some algebra,

$$\sum_{\text{odd } n\geq 3} n e^{-\pi(n^2-1)\sqrt{3}/8} = \sum_{m\geq 1} (2m+1) e^{-\pi(m^2+m)\sqrt{3}/2}.$$

Since $m^2 + m \geq 2m$,

$$\sum_{m\geq 1} (2m+1) e^{-\pi(m^2+m)\sqrt{3}/2} \leq \sum_{m\geq 1} (2m+1) e^{-\pi m\sqrt{3}}.$$

This upper bound is a power series in $e^{-\pi\sqrt{3}} \approx .0043$. For $0 < x < 1$, set

$$f(x) = \sum_{m\geq 1}(2m+1)x^m = 2\sum_{m\geq 1} mx^m + \sum_{m\geq 1} x^m = \frac{2x}{(1-x)^2} + \frac{x}{1-x}.$$

This rational function is strictly increasing on $(0,1)$ (its derivative is $(3+x)/(1-x)^3$). The unique number in $(0,1)$ where f has value 1 is $(5-\sqrt{17})/4 \approx .218$, which is greater than $e^{-\pi\sqrt{3}} \approx .0043$, so $f(e^{-\pi\sqrt{3}}) < 1$, and therefore the sum we care about is also less than 1. More precisely, $f(e^{-\pi\sqrt{3}}) \approx .01309$, so the sum we care about is less than .01309. $\square$

Our method of proving finite-dimensionality of the spaces M_k depended in a crucial way on the existence of a modular form that is nonvanishing on $\mathfrak{h}$ with a simple 0 at $i\infty$. The modular forms of positive weight for groups other than $\mathrm{SL}_2(\mathbf{Z})$ do not typically include a form that is nowhere zero on $\mathfrak{h}$, so proving finite-dimensionality of spaces of modular forms in general requires more sophisticated ideas, such as the Riemann-Roch theorem.

Exercises.

1. From $E_8 = E_4^2$ deduce for $n \geq 1$ that $\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m)$.
2. Let $f \in M_k$ and $g \in M_\ell$. Show $kf(\tau)g'(\tau) - \ell f'(\tau)g(\tau) \in M_{k+\ell+2}$, where the differentiation is with respect to τ. This is a special case of a more general bilinear operation $M_k \times M_\ell \to M_{k+\ell+2n}$ for each $n \geq 0$ called the nth Rankin–Cohen bracket. When $n = 0$ it is ordinary multiplication of modular forms, and when $n = 1$ it is essentially the operation described in this exercise. (Hint: Start by taking the derivative with respect to τ of both sides of the modularity condition. Do not confuse $(f((a\tau+b)/(c\tau+d)))'$ with $f'((a\tau+b)/(c\tau+d))$.)
3. Show the ratio E_6/E_4 satisfies the modularity condition for weight 2. Why doesn't this contradict $M_2 = \{0\}$?
4. If $f \in M_k$ is nonvanishing on $\mathfrak{h}$ then prove $12 \mid k$ and f equals $\Delta^{k/12}$ up to multiplication by a nonzero constant.
5. (a) Let $f\colon \mathbf{R} \to \mathbf{C}$ be an absolutely integrable function. For $a > 0$ and $b \in \mathbf{R}$, set $f_{a,b}(x) = f(ax+b)$. Prove the Fourier transform of $f_{a,b}$ is
$$\widehat{f_{a,b}}(y) = \frac{e^{2\pi i b/a}}{a}\widehat{f}\left(\frac{y}{a}\right).$$
   (b) Prove $xe^{-\pi x^2}$ has Fourier transform $-iye^{\pi y^2}$ and then use part (a) to find the Fourier transform of $xe^{-\pi a x^2}$ for $a > 0$.
   (c) Prove (5.2). (Hint: Write $\sum_{\text{odd } n}(-1)^{(n-1)/2}f(n)$ as
$$\sum_{m\in\mathbf{Z}} f(4m+1) - \sum_{m\in\mathbf{Z}} f(4m-1)$$
   and apply the usual Poisson summation formula to the functions $f(4x+1)$ and $f(4x-1)$, whose Fourier transforms are described by part (a).)
6. For $\theta(\tau) = \sum_{\text{odd } n\geq 1}(-1)^{(n-1)/2}ne^{\pi i n^2\tau/4}$, show $\theta(\tau+1) = \frac{1+i}{\sqrt{2}}\theta(\tau)$.
7. Use the fact that $f(x) = e^{-\pi a x^2}$ has Fourier transform $\widehat{f}(y) = (1/\sqrt{a})e^{\pi y^2/a}$ to prove that the function $\widetilde{\theta}(\tau) := \sum_{n\in\mathbf{Z}} e^{\pi i n^2\tau} = 1 + \sum_{n\geq 1} 2e^{\pi i n^2\tau}$ satisfies $\widetilde{\theta}(-1/\tau)^4 = -\tau^2\widetilde{\theta}(\tau)^4$.

6. The Eisenstein basis

We computed $\dim M_k$ without writing down a basis (when $k > 12$). In this section we describe an explicit basis built out of Eisenstein series, and more precisely it will be built from E_4 and E_6.

How did Eisenstein series play a role leading up to Theorem 5.4? We used $E_4(i) > 0$ in the proof that $M_2 = \{0\}$, and when we showed $(f - a_0 E_k)/\Delta$ is a modular form the only property we needed of E_k is that it lies in M_k and has constant term 1. For every even $k \geq 4$ we can write $k = 4a + 6b$ for some nonnegative integers a and b, so $E_4^a E_6^b$ is in M_k with constant term 1. Therefore we can prove Theorem 5.4 using only the Eisenstein series E_4 and E_6: no E_k for $k > 6$ is needed for the proof.

Theorem 6.1. *For even $k \geq 0$, the set $\{E_4^a E_6^b : a, b \geq 0, 4a + 6b = k\}$ is a basis of M_k.*

Proof. Let N_k be the number of solutions to $4a + 6b = k$ in nonnegative integers a and b. By a direct check, $N_k = \dim M_k$ for $k \leq 12$. Since $N_k = 1 + N_{k-12}$ for $k \geq 12$, $N_k = \dim M_k$ for all k. So the proposed basis $\{E_4^a E_6^b : a, b \geq 0, 4a + 6b = k\}$ has the right size.

To show this set is linearly independent, we may suppose $k \geq 14$. Let

$$\sum_{\substack{4a+6b=k \\ a,b \geq 0}} c_{a,b} E_4(\tau)^a E_6(\tau)^b = 0$$

for all τ. If there is a pure E_4 term, say $c_{A,0} E_4(\tau)^A$, then setting $\tau = i$ shows $c_{A,0} E_4(i)^A = 0$ since $E_6(i) = 0$ (Exercise 3.3). Since $E_4(i) > 0$, $c_{A,0} = 0$. Therefore all nonzero terms in the sum have $b \geq 1$. As E_6 is not identically 0, we may divide by it and get

$$\sum c_{a,b} E_4(\tau)^a E_6(\tau)^{b-1} = 0,$$

a linear relation in weight $k - 6$. By induction the remaining coefficients are 0. □

Definition 6.2. The basis $\{E_4^a E_6^b : 4a + 6b = k\}$ of M_k will be called the *Eisenstein basis*.

The following application of the Eisenstein basis depends on E_4 and E_6 having all rational Fourier coefficients.

Theorem 6.3. *If $k > 0$ and $f \in M_k$ has q-expansion $\sum_{n \geq 0} a_n q^n$ with $a_n \in \mathbf{Q}$ for all $n \geq 1$, then $a_0 \in \mathbf{Q}$.*

Proof. Before we do anything with modular forms, we will prove a result from abstract algebra that describes rational numbers using field automorphisms of the complex numbers.

There are two known field automorphisms of $\mathbf{C}$: the identity and complex conjugation. Many additional field automorphisms of $\mathbf{C}$ exist, since Zorn's lemma (the axiom of choice) can be used to prove for any subfield $F \subset \mathbf{C}$ that any field automorphism of F can be extended (somehow, usually in many ways) to a field automorphism of $\mathbf{C}$. For a proof, see Corollary 4 of http://www.math.uconn.edu/~kconrad/blurbs/zorn2.pdf.

As an example, if $F = \mathbf{Q}(\sqrt{2})$ then the automorphism $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ on F extends (in infinitely many ways in fact) to an automorphism of $\mathbf{C}$. Such an extension is neither the identity nor complex conjugation, since the extension does not fix $\sqrt{2}$ but the identity and complex conjugation both fix $\sqrt{2}$. No automorphism of $\mathbf{C}$ besides the identity or complex conjugation is continuous, and the extra field automorphisms can't be written down using explicit formulas, so their existence really needs Zorn's lemma.

Field automorphisms of $\mathbf{C}$ can tell us whether or not a complex number is rational.

Claim: If $a \in \mathbf{C}$ is not rational then there is a field automorphism $\sigma\colon \mathbf{C} \to \mathbf{C}$ that does not fix a.

Proof of claim: We take cases depending on if a is algebraic or transcendental over $\mathbf{Q}$. If a is algebraic over $\mathbf{Q}$ and $a \notin \mathbf{Q}$, let F be the splitting field of $\mathbf{Q}(a)$ over $\mathbf{Q}$. By Galois theory, there is a field automorphism of F that does not fix a. Any extension σ of this automorphism to $\mathbf{C}$ will not fix a. If a is instead transcendental over $\mathbf{Q}$, let $F = \mathbf{Q}(a)$. Then F is isomorphic to the rational function field $\mathbf{Q}(X)$ for an indeterminate X, so $a \mapsto 1/a$ (or $a \mapsto -a$) defines a field automorphism of F not fixing a. This automorphism of F extends to an automorphism of $\mathbf{C}$ and does not fix a. This concludes the proof of the claim.

Now we turn to the part of the proof that involves modular forms.

Each modular form for $\mathrm{SL}_2(\mathbf{Z})$ is determined by its q-expansion, so we can embed the vector space M_k into the ring of *formal* power series $\mathbf{C}[[q]]$ by thinking about each modular form as its q-expansion

$$\sum_{n\geq 0} a_n q^n, \qquad a_n \in \mathbf{C},$$

viewed purely formally in $\mathbf{C}[[q]]$. For example, the two Eisenstein series E_4 and E_6 are viewed as series in $\mathbf{C}[[q]]$ that both have all coefficients in $\mathbf{Q}$.

For any field automorphism σ of $\mathbf{C}$ we can define a ring automorphism r_σ of $\mathbf{C}[[q]]$ by mapping every formal power series $\sum a_n q^n$ to the formal power series $\sum \sigma(a_n) q^n$. If $f = \sum_{n\geq 1} a_n q^n$ is in M_k, is $r_\sigma(f) = \sum \sigma(a_n) q^n$ the q-expansion of a modular form?

Yes! To prove this, we can assume k is even and at least 4, since otherwise M_k is $\{0\}$ or $\mathbf{C}$. Write f as a $\mathbf{C}$-linear combination of the Eisenstein basis for M_k:

$$f = \sum_{4a+6b=k} c_{ab} E_4^a E_6^b$$

for some complex numbers c_{ab}. Viewing both sides in $\mathbf{C}[[q]]$ and applying r_σ to this equation,

$$r_\sigma(f) = r_\sigma\left(\sum_{4a+6b=k} c_{ab} E_4^a E_6^b\right) = \sum_{4a+6b=k} \sigma(c_{ab}) r_\sigma(E_4)^a r_\sigma(E_6)^b.$$

Since the q-expansion coefficients of E_4 and E_6 are rational, $r_\sigma(E_4) = E_4$ and $r_\sigma(E_6) = E_6$. Thus

$$r_\sigma(f) = \sum_{4a+6b=k} \sigma(c_{ab}) E_4^a E_6^b.$$

This is a $\mathbf{C}$-linear combination of the q-expansions of modular forms of weight k, so $r_\sigma(f)$ is the q-expansion of a modular form of weight k (the same weight as f).

Now suppose all the q-expansions coefficients of f are in $\mathbf{Q}$ except perhaps for its constant term a_0. Then the q-expansion coefficients of f and $r_\sigma(f)$ agree everywhere except possibly in their constant terms, which are a_0 and $\sigma(a_0)$. Since f and $r_\sigma(f)$ are both in M_k, their difference $f - r_\sigma(f)$ is a constant function in M_k. The only constant function of weight $k > 0$ is 0. Therefore $r_\sigma(f) - f = 0$, so $r_\sigma(f) = f$, which implies $\sigma(a_0) = a_0$ for all automorphisms σ of $\mathbf{C}$. Thus, by the claim at the start of this proof, $a_0 \in \mathbf{Q}$. $\square$

If $\sum a_n q^n$ is the q-expansion of a modular form and $a_n \in \mathbf{Z}$ for $n \geq 1$, it is generally false that $a_0 \in \mathbf{Z}$. An example is

$$\frac{1}{240} E_4 = \frac{1}{240} + \sum_{n\geq 1} \sigma_3(n) q^n = \frac{1}{240} + q + 9q^2 + 28q^3 + \dots$$

We can now use modular forms to prove a property of the Riemann zeta-function.

Theorem 6.4. *For even $k \geq 8$, $\zeta(k)$ is a rational multiple of π^k.*

Proof. Apply Theorem 6.3 to the Eisenstein series

$$\frac{G_k(\tau)}{2(2\pi i)^k/(k-1)!} = \frac{\zeta(k)}{(2\pi i)^k/(k-1)!} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

whose q-expansion does not depend on prior knowledge of zeta-values at even integers $k \geq 8$. Since all the higher-degree Fourier coefficients $\sigma_{k-1}(n)$ are rational, the constant term is also rational, so $\zeta(k)/\pi^k$ is rational. □

The proof of Theorem 6.4 depends on Theorem 6.3, whose proof in turns depends on rationality of all the Fourier coefficients of the Eisenstein basis. The rationality of the Fourier coefficients of E_4 and E_6 requires knowing $\zeta(4)/\pi^4$ and $\zeta(6)/\pi^6$ are rational. Therefore our proof using modular forms that $\zeta(k)/\pi^k \in \mathbf{Q}$ for even integers $k \geq 8$ needs this result to be known already for $k = 4$ and $k = 6$ (the case $k = 2$ does not matter). You can check we never relied on the Eisenstein series with weight > 6 for anything but examples, so deducing the rationality of $\zeta(k)/\pi^k$ for even $k \geq 8$ from the cases $k = 4$ and 6 is not a circular argument.

This method of deducing rationality properties of zeta-values from their appearance in the constant term of a modular form can be generalized to zeta-values of all totally real number fields at positive even integers, by constructing modular forms in which the zeta-values appear in the constant term. This is the Klingen–Siegel theorem.

Are there any linear relations between forms of different weights?

Lemma 6.5. *Modular forms with different weights are linearly independent over* $\mathbf{C}$.

Proof. Let $f_1, f_2, \ldots, f_m$ be nonzero modular forms with respective weights $k_1 < k_2 < \cdots < k_m$. All weights are nonnegative. Assume the f_i satisfy a nontrivial linear relation:

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_m f_m(\tau) = 0 \tag{6.1}$$

for all $\tau \in \mathfrak{h}$, where not all α_j equal 0. We may assume this is an example with $m \geq 2$ minimal, so all α_j are nonzero.

Pick γ in $\mathrm{SL}_2(\mathbf{Z})$ with lower left entry $c \neq 0$ (i.e., $\gamma \neq \pm T^n$ for any $n \in \mathbf{Z}$). Replacing τ with $\gamma\tau$ in (6.1), the modularity condition implies

$$\alpha_1(c\tau+d)^{k_1} f_1(\tau) + \alpha_2(c\tau+d)^{k_2} f_2(\tau) + \ldots \alpha_m(c\tau+d)^{k_m} f_m(\tau) = 0 \tag{6.2}$$

for all τ.

Let $f_j(\tau)$ have q-expansion $\sum_{n\geq 0} a_n^{(j)} e^{2\pi i n\tau}$, so

$$\sum_{n\geq 0} (\alpha_1(c\tau+d)^{k_1} a_n^{(1)} + \cdots + \alpha_m(c\tau+d)^{k_m} a_n^{(m)}) e^{2\pi i n\tau} = 0.$$

Look at this along the imaginary axis: for $\tau = iy$ with $y > 0$,

$$\sum_{n\geq 0} (\alpha_1(ciy+d)^{k_1} a_n^{(1)} + \cdots + \alpha_m(ciy+d)^{k_m} a_n^{(m)}) e^{-2\pi n y} = 0. \tag{6.3}$$

For $n > 0$, $y^r e^{-2\pi n y} \to 0$ as $y \to \infty$ for any $r \geq 0$, so if we divide through (6.3) by $e^{-2\pi N y}$ for the smallest N such that some $a_N^{(j)}$ is nonzero and then let $y \to 0$, we are left with

$$\lim_{y\to\infty} \alpha_1(ciy+d)^{k_1} a_N^{(1)} + \cdots + \alpha_m(ciy+d)^{k_m} a_N^{(m)} = 0.$$

All α_j are nonzero, some $a_N^{(j)}$ is nonzero, and the weights k_j are distinct, so the left side is the limit of a nonconstant polynomial in y as $y \to \infty$. We have a contradiction. □

The proof hardly used the modularity condition for $\mathrm{SL}_2(\mathbf{Z})$; only one $\gamma \neq \pm T^n$ was needed.

Since $M_k M_\ell \subset M_{k+\ell}$, the $\mathbf{C}$-linear combinations of modular forms of all weights for $\mathrm{SL}_2(\mathbf{Z})$ is not only a vector space, but a ring containing $M_0 = \mathbf{C}$. (The sum of modular forms of different weights is is not a modular form.) By Theorem 6.1, the forms $E_4^a E_6^b$ for general $a, b \geq 0$ span the $\mathbf{C}$-algebra generated by all modular forms, so the ring generated over $\mathbf{C}$ by modular forms for $\mathrm{SL}_2(\mathbf{Z})$ is $\mathbf{C}[E_4, E_6]$.

Theorem 6.6. *The modular forms E_4 and E_6 are algebraically independent over $\mathbf{C}$.*

Proof. Suppose $P(X, Y)$ is a nonzero polynomial in $\mathbf{C}[X, Y]$ such that $P(E_4(\tau), E_6(\tau)) = 0$ for all τ. For each monomial $a_{ij} X^i Y^j$ in $P(X, Y)$, the function $a_{ij} E_4^i E_6^j$ is a modular form of weight $4i+6j$, so if we collect together in $P(E_4, E_6)$ all monomial terms of a common weight then Lemma 6.5 gives us equations $Q_k(E_4, E_6) = 0$ where each monomial term appearing in here has the same weight k. By Theorem 6.1 the coefficients occurring in $Q_k(X, Y)$ (which are all coefficients from $P(X, Y)$) are all 0, so $P = 0$. □

Corollary 6.7. *The ring generated over $\mathbf{C}$ by all modular forms for $\mathrm{SL}_2(\mathbf{Z})$ is isomorphic to the polynomial ring $\mathbf{C}[X, Y]$.*

Proof. The ring is $\mathbf{C}[E_4, E_6]$, so the algebraic independence of E_4 and E_6 over $\mathbf{C}$ implies $\mathbf{C}[E_4, E_6] \cong \mathbf{C}[X, Y]$. □

Exercises.

1. (a) Express E_{18} as a linear combinations of E_6^3 and $E_4^3 E_6$.
   (b) Express each of E_{12}^2, $E_6 E_8 E_{10}$, and E_{24} as linear combinations of E_4^6, $E_4^3 E_6^2$, and E_6^4.

7. Modular forms in terms of $\mathrm{SL}_2(\mathbf{Z})$-invariant functions

Let $f \colon \mathfrak{h} \to \mathbf{C}$ be a modular form of weight k for $\mathrm{SL}_2(\mathbf{Z})$. We saw in the proof of Theorem 5.1 that the real-valued function $|f(\tau)|(\operatorname{Im} \tau)^{k/2}$ is $\mathrm{SL}_2(\mathbf{Z})$-invariant (the theorem was concerned with $k < 0$, but this part of the proof goes through for all integers k). Although it might seem at first that taking absolute values on f destroys some information about f being a modular form, we can nearly recover the modularity condition from the $\mathrm{SL}_2(\mathbf{Z})$-invariance of $|f(\tau)|(\operatorname{Im} \tau)^{k/2}$ if we remember f is holomorphic. Here is a general version of this type of result.

Theorem 7.1. *Let Γ be a subgroup of $\mathrm{SL}_2(\mathbf{R})$. If $f \colon \mathfrak{h} \to \mathbf{C}$ is holomorphic then the following conditions are equivalent:*

(1) *$|f(\tau)|(\operatorname{Im} \tau)^{k/2}$ is a Γ-invariant function: $|f(\gamma\tau)|(\operatorname{Im} \gamma\tau)^{k/2} = |f(\tau)|(\operatorname{Im} \tau)^{k/2}$ for all $\gamma \in \Gamma$ and $\tau \in \mathfrak{h}$,*
(2) *there is a group homomorphism $\chi \colon \Gamma \to S^1$ such that $f(\gamma\tau) = \chi(\gamma)(c_\gamma \tau + d_\gamma)^k f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} * & * \\ c_\gamma & d_\gamma \end{smallmatrix}\right)$ in Γ.*

Proof. That (2) implies (1) follows by the same reasoning as in the proof of Theorem 5.1 since $|\chi(\gamma)| = 1$ for all $\gamma \in \Gamma$.

To prove (1) implies (2), this is obvious if f is identically 0 (let χ be any homomorphism, even the trivial one), so we can assume f is not identically 0. Then for each $\gamma \in \Gamma$,

$$\begin{aligned} |f(\gamma\tau)|(\operatorname{Im}\gamma\tau)^{k/2} = |f(\tau)|(\operatorname{Im}\tau)^{k/2} &\iff |f(\gamma\tau)| = \left(\frac{\operatorname{Im}\tau}{\operatorname{Im}\gamma\tau}\right)^{k/2}|f(\tau)| \\ &\iff |f(\gamma\tau)| = \left(\frac{\operatorname{Im}\tau}{(\operatorname{Im}\tau)/|c_\gamma\tau+d_\gamma|^2}\right)^{k/2}|f(\tau)| \\ &\iff |f(\gamma\tau)| = |c_\gamma\tau+d_\gamma|^k|f(\tau)|, \end{aligned}$$

so $f(\gamma\tau)$ and $g(\tau) := (c_\gamma\tau+d_\gamma)^kf(\tau)$ are both holomorphic in τ and are not identically 0, and $|f(\gamma\tau)| = |g(\tau)|$ for all $\tau \in \mathfrak{h}$. We want to show there is some number $\alpha \in S^1$ such that $f(\gamma\tau) = \alpha g(\tau)$ for all τ.

There is a ball B in $\mathfrak{h}$ on which $f(\tau)$ is nonvanishing. Then $f(\gamma\tau)$ and $g(\tau)$ are nonvanishing on B, so their ratio $f(\gamma\tau)/g(\tau)$ is holomorphic on B and has values in the unit circle. The Open Mapping Theorem from complex analysis says a nonconstant holomorphic function on a connected open subset Ω of $\mathbf{C}$ sends open subsets to open subsets. In particular, a holomorphic function with values in S^1 must be constant, so there is a number $\alpha \in S^1$ such that $f(\gamma\tau)/g(\tau) = \alpha$ on B. Then $f(\gamma\tau) = \alpha g(\tau)$ on B, so by rigidity of holomorphic functions on $\mathfrak{h}$ we get $f(\gamma\tau) = \alpha g(\tau)$ for all $\tau \in \mathfrak{h}$.

The constant α depends on the choice of γ, so write it as $\chi(\gamma)$: for each $\gamma \in \Gamma$ we showed there is some $\chi(\gamma) \in S^1$ such that $f(\gamma\tau) = \chi(\gamma)g(\tau) = \chi(\gamma)(c_\gamma\tau+d_\gamma)^kf(\tau)$ for all $\tau \in \mathfrak{h}$. Why is $\chi\colon \Gamma \to S^1$ a homomorphism? For $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ in Γ and any $\tau \in \mathfrak{h}$, $\gamma_1\gamma_2$ has second row entries $c_1a_2+d_1c_2$ and $c_1b_2+d_1d_2$ so $f((\gamma_1\gamma_2)\tau) = \chi(\gamma_1\gamma_2)((c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2)\tau)^kf(\tau)$, and also

$$\begin{aligned} f((\gamma_1\gamma_2)\tau) &= f(\gamma_1(\gamma_2\tau)) \\ &= \chi(\gamma_1)(c_1(\gamma_2\tau)+d_1)^kf(\gamma_2\tau) \\ &= \chi(\gamma_1)\left(c_1\left(\frac{a_2\tau+b_2}{c_2\tau+d_2}\right)+d_1\right)^k\chi(\gamma_2)(c_2\tau+d_2)^kf(\tau) \\ &= \chi(\gamma_1)\chi(\gamma_2)\frac{(c_1(a_2\tau+b_2)+d_1(c_2\tau+d_2))^k}{(c_2\tau+d_2)^k}(c_2\tau+d_2)^kf(\tau) \\ &= \chi(\gamma_1)\chi(\gamma_2)((c_1a_2+d_1c_2)\tau+(c_1b_2+d_1d_2))^kf(\tau). \end{aligned}$$

The two expressions we found for $f(\gamma_1\gamma_2\tau)$ are exactly the same except for the factors $\chi(\gamma_1\gamma_2)$ and $\chi(\gamma_1)\chi(\gamma_2)$, so (since f is not identically 0) we get $\chi(\gamma_1\gamma_2) = \chi(\gamma_1)\chi(\gamma_2)$. □

This theorem suggests generalizing the concept of modular forms to allow modular forms "with character." For example, if $\chi\colon \mathrm{SL}_2(\mathbf{Z}) \to S^1$ is a homomorphism (a one-dimensional character) then a modular form of weight k with character χ for $\mathrm{SL}_2(\mathbf{Z})$ would be a holomorphic function $f\colon \mathfrak{h} \to \mathbf{C}$ that satisfies the condition

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(c\tau+d)^kf(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ and is bounded as $\tau \to i\infty$. A modular form in its original definition would be a modular form with trivial character. Theorem 7.1 tells us a modular form of weight k with some character is a holomorphic function $f\colon \mathfrak{h} \to \mathbf{C}$ bounded as $\tau \to i\infty$ such that $|f(\tau)|(\operatorname{Im}\tau)^{k/2}$ is an $\mathrm{SL}_2(\mathbf{Z})$-invariant function.

Homomorphisms $\mathrm{SL}_2(\mathbf{Z}) \to S^1$ are trivial on the commutator subgroup $[\mathrm{SL}_2(\mathbf{Z}), \mathrm{SL}_2(\mathbf{Z})]$, which turns out to be a subgroup of index 12, and there are 12 characters on $\mathrm{SL}_2(\mathbf{Z})$.

8. Coefficient estimates in q-expansions

When we write a modular form $f \in M_k$ as a q-expansion $\sum_{n\geq 0} a_n q^n$, how quickly do the coefficients grow? Even a cursory glance at the coefficients of Eisenstein series shows they seem to get large (in absolute value) quickly. We will prove in this section an upper bound on $|a_n|$, showing it grows no faster than a simple power of n depending on the weight of f.

Example 8.1. For the Eisenstein series $E_k = 1-(2k/B_k)\sum_{n\geq 1}\sigma_{k-1}(n)q^n$, the nth Fourier coefficient is $(-2k/B_k)\sigma_{k-1}(n)$. Since $k \geq 4$, the divisor sum $\sigma_{k-1}(n)$ grows no faster than a constant multiple of n^{k-1}:

$$\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1} = n^{k-1}\sum_{d\mid n}\frac{1}{(n/d)^{k-1}} \leq n^{k-1}\sum_{m\geq 1}\frac{1}{m^{k-1}} = \zeta(k-1)n^{k-1}$$

and trivially $\sigma_{k-1}(n) \geq n^{k-1}$, so the nth Fourier coefficient of E_k grows like n^{k-1} as $n \to \infty$, to within constant multiples above and below.

To expand this example to an estimate on $|a_n|$ in the general case, we need to focus attention on the modular forms that vanish at $i\infty$.

Definition 8.2. A modular form $f \in M_k$ is called a *cusp form* if the constant term of its q-expansion is 0. The set of all cusp forms of weight k is denoted S_k.

The letter S in S_k is taken from the word Spitzenform which is German for cusp form: Spitze means "cusp" (or "tip, spike") in German.[9] The reason for this terminology is that for modular forms on groups other than $\mathrm{SL}_2(\mathbf{Z})$ the notion of a cusp form includes vanishing conditions at points besides $i\infty$ where the fundamental domain for the group touches the boundary of the upper half-plane, and the shape of the fundamental domain near those boundary points looks like a cusp (see the fundamental domain for $\Gamma_0(2)$ near 0).

The space S_k of cusp forms in M_k is the kernel of the linear map $M_k \to \mathbf{C}$ given by evaluating modular forms at $i\infty$. Thus $M_k/S_k \cong \mathbf{C}$ when $M_k \neq \{0\}$, so $\dim S_k = \dim M_k - 1$ when $M_k \neq \{0\}$. In particular, $S_k \neq \{0\}$ if and only if $\dim M_k \geq 2$, so the first k where $S_k \neq 0$ is $k = 12$: $\dim S_{12} = \dim M_{12} - 1 = 2 - 1 = 1$.

Example 8.3. The modular form $\Delta(\tau) = q + \cdots$ is a cusp form in S_{12}. Since S_{12} is 1-dimensional, any two methods of constructing a cusp form of weight 12 for $\mathrm{SL}_2(\mathbf{Z})$ will lead to the same function to within a constant multiple.

Unlike Eisenstein series, whose Fourier coefficients have explicit formulas, the Fourier coefficients of cusp forms usually do not admit simple general formulas and their size is much smaller than those of Eisenstein series.

Theorem 8.4. *If $f = \sum_{n\geq 1} a_n q^n$ is a cusp form of weight k for $\mathrm{SL}_2(\mathbf{Z})$ then $a_n = O(n^{k/2})$.*

Proof. We use an idea from the proof that the only modular form of negative weight is zero (Theorem 5.1). In that proof we showed $|f(\tau)|(\mathrm{Im}\,\tau)^{k/2}$ is an $\mathrm{SL}_2(\mathbf{Z})$-invariant function no matter what the weight is (positive, negative, or 0).

[9] In French the term is forme parabolique, and the Russian term is similar to this, because cusps are also called parabolic points.

When $f \in S_k$, so its q-expansion starts out as $a_1q + \cdots$ (the coefficient a_1 may or may not be 0), then as a function on the open unit disc we can say $|f(q)| = O(|q|)$ as $q \to 0$. Therefore if $\tau = x + iy$ and $y \to \infty$ we have $|f(\tau)| = O(e^{-2\pi y})$ as $y \to \infty$. Thus $|f(\tau)|y^{k/2} = O(e^{-2\pi y}y^{k/2}) \to 0$ as $y \to \infty$, and the boundedness of $|f(\tau)|(\operatorname{Im}\tau)^{k/2}$ on $\mathfrak{h}$ now follows just as in the proof of Theorem 5.1.

Letting $|f(\tau)|(\operatorname{Im}\tau)^{k/2} \leq C$ for all τ, we get

$$|a_n| \leq Ce^{2\pi ny}y^{-k/2} \tag{8.1}$$

for all $y > 0$ as in the proof of Theorem 5.1. In that proof we let $y \to 0^+$ to show $a_n = 0$ since $k < 0$, but for $k \geq 0$ we don't get progress by letting $y \to 0^+$. Instead, simply set $y = 1/n$ to see that $|a_n| \leq Ce^{2\pi}n^{k/2} = O(n^{k/2})$. □

Theorem 8.5. *For even $k \geq 4$ and $f = \sum_{n\geq 0} a_nq^n$ in M_k, $a_n = O(n^{k-1})$, and a_n grows like n^{k-1} to within a constant multiple if and only if f is not a cusp form.*

Proof. We know by Theorem 8.4 that $a_n = O(n^{k/2})$ if f is a cusp form, so it remains to show if f is not a cusp form that $An^{k-1} \leq |a_n| \leq Bn^{k-1}$ for all n and some constants A and B (depending perhaps on k).

Both f and a_0E_k are in M_k with constant term a_0, so the difference $g := f - a_0E_k$ is a cusp form of weight k. Letting $g = \sum_{n\geq 0} b_nq^n$, we have

$$a_n = a_0\left(-\frac{2k}{B_k}\right)\sigma_{k-1}(n) + b_n.$$

Since $a_0 \neq 0$ the first term grows like n^{k-1} to within constant multiples while the second term grows at most like $n^{k/2}$. □

9. Modular Forms and Dirichlet Series

A *Dirichlet series* is an infinite series of the form

$$\sum_{n\geq 1} \frac{a_n}{n^s}.$$

For example, if $a_n = 1$ for all n then this Dirichlet series is the Riemann zeta-function $\zeta(s) = \sum_{n\geq 1} 1/n^s$, which is absolutely convergent when $\operatorname{Re}(s) > 1$ because

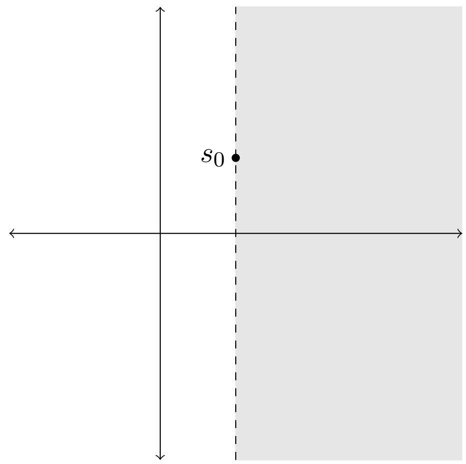
$$\sum_{n\geq 1}\left|\frac{1}{n^s}\right| = \sum_{n\geq 1}\frac{1}{n^{\operatorname{Re}(s)}} < \infty$$

when $\operatorname{Re}(s) > 1$ by the integral test from calculus. Dirichlet series are not an all-purpose tool like power series in analysis, but they are very important series in number theory.

The convergence properties of Dirichlet series are both similar to and different from power series. For example, if a power series $\sum_{n\geq 0} c_nz^n$ converges at a number z_0 then in the disc $\{z : |z| < |z_0|\}$ the power series is absolutely convergent and also uniformly convergent on compact subsets of that disc, which justifies termwise differentiation of the power series. If a Dirichlet series converges at a number s_0 then in the right half-plane $\{s : \operatorname{Re}(s) > \operatorname{Re}(s_0)\}$, which is pictured below, the series is convergent[10] and is also uniformly convergent on compact subsets of the half-plane $\{s : \operatorname{Re}(s) > \operatorname{Re}(s_0)\}$, which implies the Dirichlet series is

[10] Absolute convergence at s_0 implies absolute convergence for $\operatorname{Re}(s) > \operatorname{Re}(s_0)$, but just having convergence at s_0 only implies absolute convergence for $\operatorname{Re}(s) > \operatorname{Re}(s_0) + 1$. For example, the "alternating zeta-function" $\sum_{n\geq 1}(-1)^{n-1}/n^s$ converges for $\operatorname{Re}(s) > 0$ but is absolutely convergent only for $\operatorname{Re}(s) > 1$.

holomorphic for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$ and can be differentiated termwise there. Proofs of these convergence properties of Dirichlet series can be found in analytic number theory textbooks.



Although the Dirichlet series defining $\zeta(s)$ only converges for $\mathrm{Re}(s) > 1$, Riemann used other formulas for the zeta-function to show $\zeta(s)$ has an analytic continuation to $\mathbf{C}$ except for a simple pole at $s = 1$ (with residue 1) and there is a functional equation relating $\zeta(s)$ to $\zeta(1-s)$:

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s), \tag{9.1}$$

where the Gamma-function $\Gamma(s)$ is the meromorphic function on $\mathbf{C}$ defined for $\mathrm{Re}(s) > 0$ as the integral $\int_0^\infty t^s e^{-s}\,dt/t$ and continued to the rest of $\mathbf{C}$ by the formula $\Gamma(s+1) = s\Gamma(s)$ (proved for $\mathrm{Re}(s) > 0$ with integration by parts). As an example of the functional equation, taking $s = 2$ we get

$$\zeta(-1) = \frac{2}{4\pi^2}\cos(\pi)\Gamma(2)\zeta(2) = -\frac{1}{2\pi^2}\frac{\pi^2}{6} = -\frac{1}{12}.$$

(For some other s one needs to be careful about cancellation of zeros and poles in different factors of (9.1) in order to evaluate the right side at s: at $s = 1$ there is a simple zero in $\cos(\pi s/2)$ and a simple pole in $\zeta(s)$, which cancel out and leave $\zeta(0) = -1/2$.) Riemann found (9.1) is equivalent to a cleaner functional equation for the "completed zeta-function" $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$:

$$Z(1-s) = Z(s). \tag{9.2}$$

Riemann's proof of the analytic continuation of $\zeta(s)$ or $Z(s)$ used the Jacobi theta-function $\Theta(y) = \sum_{n\in\mathbf{Z}} e^{-\pi n^2 y}$. He showed for $\mathrm{Re}(s) > 1$ that

$$Z(s) = \frac{1}{2}\int_0^\infty (\Theta(y) - 1)y^{s/2}\,\frac{dy}{y},$$

then split up the integral into two integrals over $[0,1]$ and $[1,\infty)$, rewrote the integral over $[0,1]$ as an integral over $[1,\infty)$, and used the transformation law $\Theta(1/y) = \sqrt{y}\Theta(y)$ to put the integrals in a form that were visibly unchanged when s is replaced with $1-s$, which is the functional equation (9.2). The transformation law for $\Theta(1/y)$, involving a square root of y, reflects that $\Theta(y)$ is closely related to a modular form of weight $1/2$. Nowadays many years after Riemann, we know many Dirichlet series besides $\zeta(s)$ that converge on some right half-plane and have a meromorphic continuation to $\mathbf{C}$ that satisfies a functional equation like (9.2) after the Dirichlet series is multiplied by exponential and Γ-functions like $\zeta(s)$ is to form $Z(s)$. Essentially the only known method of proving the meromorphic continuation and functional equation is to connect the coefficients of the Dirichlet series to something like a modular form. In this section, we illustrate that connection by showing how to associate a Dirichlet series directly to a modular form and use properties of the modular form to get the meromorphic continuation and functional equation for the Dirichlet series.

Definition 9.1. The *L-function* of a modular form $f \in M_k$ with q-expansion $\sum_{n\geq 0} a_n q^n$ is $L(f,s) = \sum_{n\geq 1} a_n/n^s$.

The coefficient a_0 does not appear in $L(f,s)$, but we'll see how $L(f,s)$, as an analytic function, knows what a_0 is. Our first order of business is to say where $L(f,s)$ converges.

Theorem 9.2. *The series $L(f,s)$ is absolutely convergent for $\mathrm{Re}(s) > k$, where k is the weight of f.*

Proof. Let f have q-expansion $\sum_{n\geq 0} a_n q^n$. Then $|a_n| \leq C_k n^{k-1}$ for some constant C_k, by Theorem 8.5, so

$$\sum_{n\geq 1}\left|\frac{a_n}{n^s}\right| \leq \sum_{n\geq 1} C_k \frac{1}{n^{\mathrm{Re}(s)-(k-1)}} = C_k \sum_{n\geq 1} \frac{1}{n^{\mathrm{Re}(s)-k+1}}.$$

For $\mathrm{Re}(s) > k$ the exponent on n is greater than 1, so the series is absolutely convergent. $\square$

Example 9.3. For even $k \geq 4$, the L-function of E_k is essentially a product of two zeta-functions:

$$L(E_k,s) = -\frac{2k}{B_k}\sum_{n\geq 1}\frac{\sigma_{k-1}(n)}{n^s} = -\frac{2k}{B_k}\zeta(s)\zeta(s-k+1).$$

Remark 9.4. Theorem 9.2 is not saying $L(f,s)$ is never absolutely convergent outside $\mathrm{Re}(s) > k$. For example, if f is a cusp form of weight k then its coefficients grow at most like $O(n^{k/2})$ by Theorem 8.4, so $L(f,s)$ converges absolutely for $\mathrm{Re}(s) > k/2+1$. If the $O(n^{k/2})$-estimate can be sharpened then the half-plane of absolute convergence of $L(f,s)$ would become even larger (there are results in this direction: the Ramanujan-Petersson conjecture, proved by Deligne).

Theorem 9.5. *For even $k \geq 4$ and $f \in M_k$, the "completed L-function" of f*

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$$

has an analytic continuation to $\mathbf{C}$ except for at worst simple poles at $s=0$ and $s=k$ with residues

$$\mathrm{Res}_{s=0}\,\Lambda(f,s) = -a_0, \quad \mathrm{Res}_{s=k}\,\Lambda(f,s) = (-1)^{k/2}a_0,$$

where a_0 is the constant term of the q-expansion of f. The function $\Lambda(f,s)$ satisfies the functional equation

$$\Lambda(f,k-s) = (-1)^{k/2}\Lambda(f,s).$$

This theorem shows how $L(f,s)$ "knows" the constant term a_0 of f even though a_0 is not a coefficient in this Dirichlet series: it appears in the residue of $(2\pi)^{-s}\Gamma(s)L(f,s)$ at 0.

If $a_0 = 0$, namely f is a cusp form, then the residues at $s=0$ and $s=k$ both vanish so the theorem is saying $\Lambda(f,s)$ is an entire function.

Proof. Here is the plan of the proof. For $s \in \mathbf{C}$ having $\mathrm{Re}(s) > k$, $L(f,s)$ is already defined. We will express $\Lambda(f,s)$ for such s as an integral over $[0,\infty)$, break up the integral into a sum of integrals over $[0,1]$ and $[1,\infty)$, and convert the integral over $[0,1]$ into an integral over $[1,\infty)$ by a change of variables. The two integrals over $[1,\infty)$ will each make sense for all $s \in \mathbf{C}$ and provide the analytic continuation of $\Lambda(f,s)$ to all of $\mathbf{C}$. The modularity condition for $f(-1/\tau)$, along the positive imaginary axis, will be used to combine the two integrals over $[1,\infty)$ into a single integral that has the desired symmetry under the substitution $s \mapsto k - s$. Along the way we will pick up polar terms $-a_0/s$ and $(-1)^{k/2}a_0/(s-k)$.

Fix $s \in \mathbf{C}$ with $\mathrm{Re}(s) > k$. Then

$$\begin{aligned}(2\pi)^{-s}\Gamma(s)L(f,s) &= (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1}\frac{a_n}{n^s}\\ &= \sum_{n\geq 1}\frac{a_n}{(2\pi n)^s}\Gamma(s)\\ &= \sum_{n\geq 1}\frac{a_n}{(2\pi n)^s}\int_0^\infty t^s e^{-t}\frac{dt}{t}\\ &= \sum_{n\geq 1}a_n\int_0^\infty\left(\frac{t}{2\pi n}\right)^s e^{-t}\frac{dt}{t}.\end{aligned}$$

In the integral, make the change of variables $y = t/2\pi n$, so $dt/t = dy/y$ and

$$(2\pi)^{-s}\Gamma(s)L(f,s) = \sum_{n\geq 1}a_n\int_0^\infty y^s e^{-2\pi n y}\frac{dy}{y}.$$

The series on the right converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s) > k+1$, so we can interchange the sum and integral:

$$\Lambda(f,s) = \int_0^\infty\left(\sum_{n\geq 1}a_n e^{-2\pi n y}\right)y^s\frac{dy}{y}.$$

The series inside the integral is $f(iy)$ without its constant term, so $f(iy) - a_0$. Thus

$$\Lambda(f,s) = \int_0^\infty (f(iy)-a_0)y^s\frac{dy}{y} = \int_0^1 (f(iy)-a_0)y^s\,\frac{dy}{y} + \int_1^\infty (f(iy)-a_0)y^s\,\frac{dy}{y}.$$

For $y \geq 1$, $f(iy) - a_0 = O(e^{-2\pi y})$, so the integral over $[1,\infty)$ (this is key – keep the integral bounded away from the point $y = 0$) converges for *all* $s \in \mathbf{C}$ and can be proved to be holomorphic in s. In the integral over $[0,1]$, the term $-\int_0^1 a_0 y^s dy/y$ is $-a_0/s$. In $\int_0^1 f(iy)y^s\,dy/y$, make the change of variables $y \mapsto 1/y$ to convert the integral into one over $[1,\infty)$. Since $f(i/y) = f(-1/iy) = (iy)^k f(iy) = (-1)^{k/2}y^k f(iy)$,

$$\begin{aligned}\Lambda(f,s) &= \int_0^1 f(iy)y^s\,\frac{dy}{y} - \frac{a_0}{s} + \int_1^\infty (f(iy)-a_0)y^s\,\frac{dy}{y}\\ &= \int_\infty^1 f(i/y)y^{-s}\left(-\frac{dy}{y}\right) - \frac{a_0}{s} + \int_1^\infty (f(iy)-a_0)y^s\,\frac{dy}{y}\\ &= \int_1^\infty (-1)^{k/2}f(iy)y^{k-s}\,\frac{dy}{y} - \frac{a_0}{s} + \int_1^\infty (f(iy)-a_0)y^s\,\frac{dy}{y}. \qquad (9.3)\end{aligned}$$

In this last formula, the first integral makes sense since $f(iy)$ is bounded as $y \to \infty$ and $|y^{k-s}/y| = 1/y^{\mathrm{Re}(s)-k+1}$ has exponent greater than 1 in the denominator. Since $\int_1^\infty y^{k-s}\,dy/y = 1/(k-s)$ when $\mathrm{Re}(s) > k$, we can add and subtract $a_0 y^{k-s}/y$ in the first integral:

$$\begin{aligned}\int_1^\infty (-1)^{k/2} f(iy) y^{k-s}\,\frac{dy}{y} &= (-1)^{k/2}\int_1^\infty (f(iy)-a_0)y^{k-s}\,\frac{dy}{y} + (-1)^{k/2}\int_1^\infty a_0 y^{k-s}\,\frac{dy}{y}\\ &= (-1)^{k/2}\int_1^\infty (f(iy)-a_0)y^{k-s}\,\frac{dy}{y} + (-1)^{k/2}\frac{a_0}{s-k}. \qquad (9.4)\end{aligned}$$

Since $|f(iy) - a_0| \leq Ce^{-2\pi y}$ for $y \geq 1$ and some constant C, this last integral over $[1, \infty)$ converges for all s and is entire. Feeding (9.4) into (9.3),

$$\Lambda(f, s) = \int_1^\infty (f(iy) - a_0)(y^s + (-1)^{k/2}y^{k-s})\frac{dy}{y} - \frac{a_0}{s} + (-1)^{k/2}\frac{a_0}{s-k}.$$

The integral here is entire and the two other terms provide at worst simple poles (they are not poles if $a_0 = 0$) at $s = 0$ with residue $-a_0$ and at $s = k$ with residue $(-1)^{k/2}a_0$. We can use this final expression to define $\Lambda(f, s)$ for all $s \in \mathbf{C}$.

If we replace s with $k - s$, the formula changes only by an overall factor of $(-1)^{k/2}$, so $\Lambda(f, k - s) = (-1)^{k/2}\Lambda(f, s)$. □

More can be proved about $\Lambda(f, s)$ than we have done here: it is bounded in vertical strips if we ignore small discs around the two poles in case they exist and lie in the strips.

Remark 9.6. The q-expansion of a modular form f makes essential use of the modularity condition $f(\tau + 1) = f(\tau)$, but keeps the modularity condition $f(-1/\tau) = \tau^k f(\tau)$ obscure. In the proof of the meromorphic continuation of $\Lambda(f, s)$ to $\mathbf{C}$ we made essential use of the second modularity condition.

Corollary 9.7. *For even $k \geq 4$ and $f \in M_k$, the Dirichlet series $L(f, s)$ has an analytic continuation to $\mathbf{C}$ except for a simple pole at $s = k$ if f is not a cusp form.*

Proof. In the equation $\Lambda(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s)$ solve for $L(f, s)$:

$$L(f, s) = \frac{(2\pi)^s}{\Gamma(s)}\Lambda(f, s).$$

On the right side, $1/\Gamma(s)$ is an entire function with simple zeros at integers $s \leq 0$, and $\Lambda(f, s)$ is an entire function if f is a cusp form and is entire except for simple poles at $s = 0$ and $s = k$ if f is *not* a cusp form. Therefore if f is a cusp form, $L(f, s)$ is entire. If f is not a cusp form, the simple pole of $\Lambda(f, s)$ at 0 is canceled by the simple zero of $1/\Gamma(s)$ at $s = 0$, but the simple pole of $\Lambda(f, s)$ at $s = k$ is not canceled since $\Gamma(k) = 1/(k-1)!$. □

Hecke (in 1936) proved a converse to Theorem 9.5 and its corollary. We will state the version without poles: if a Dirichlet series $D(s) = \sum_{n\geq 1} a_n/n^s$ converges in some right half-plane and the function $\mathbf{D}(s) := (2\pi)^{-s}\Gamma(s)D(s)$ extends to an entire function that is bounded in vertical strips and satisfies

$$\mathbf{D}(k - s) = (-1)^{k/2}\mathbf{D}(s)$$

for an even integer $k \geq 0$, then $\sum_{n\geq 1} a_n e^{2\pi i n\tau}$ is a cusp form of weight k for $\mathrm{SL}_2(\mathbf{Z})$.

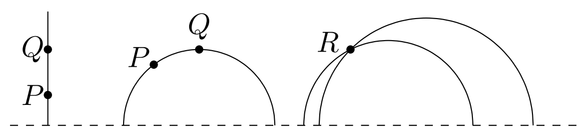
Example 9.8. Since S_{12} has dimension 1, up to scaling there is only one Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ that converges in a right half-plane and $\mathbf{D}(s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1} a_n/n^s$ extends to an entire function that is bounded in vertical strips and satisfies $\mathbf{D}(12-s) = \mathbf{D}(s)$: $\sum a_n e^{2\pi i n\tau}$ is in S_{12} so it is a scalar multiple of $\Delta(\tau)$.

Thirty years after Hecke proved his converse theorem, Weil generalized it to characterize modular forms on certain finite-index subgroups of $\mathrm{SL}_2(\mathbf{Z})$ in terms of the analytic behavior of their corresponding Dirichlet series *and* the twists of these Dirichlet series by Dirichlet characters.

Appendix A. The Hyperbolic Plane

The hyperbolic plane is the upper half-plane $\mathfrak{h}$ with a definition of lines (also called geodesics) and distances that differ from the usual meaning of these notions in the Euclidean plane $\mathbf{R}^2$.

Lines in $\mathfrak{h}$ are the vertical lines in $\mathfrak{h}$ or the semicircles in $\mathfrak{h}$ that meet the x-axis in a 90-degree angle (the x-axis is the diameter of the semicircle). In the picture below, if P and Q have the same x-coordinate then the line $\overline{PQ}$ through P and Q is the part of the usual Euclidean (vertical) line through P and Q that is in $\mathfrak{h}$. If P and Q do not have the same x-coordinate then $\overline{PQ}$ is the unique Euclidean semicircle through P and Q with diameter on the x-axis.



On the right side of the picture two lines drawn through a point R not on $\overline{PQ}$ don't intersect $\overline{PQ}$. This contradicts the parallel postulate of Euclidean geometry, which says a point not on a line L has exactly one line through it that doesn't meet L. In $\mathbf{R}^2$ the parallel postulate is true, but in $\mathfrak{h}$ it is not.

The hyperbolic distance between two points P and Q in $\mathfrak{h}$ is defined using integration along $\overline{PQ}$:

$$d_H(P,Q) = \int_P^Q \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{y(t)}\,dt,$$

where the integral is taken along the hyperbolic line in $\mathfrak{h}$ from P to Q using any smooth parametrization $(x(t), y(t))$ of the segment in $\overline{PQ}$ from P to Q.

Example A.1. To compute the distance between $y_0 i$ and $y_1 i$, parametrize the vertical line between them as $(x(t), y(t)) = (0, (1-t)y_0 + ty_1)$ for $0 \leq t \leq 1$. Then

$$d_H(y_0 i, y_1 i) = \int_0^1 \frac{\sqrt{0^2 + (y_1 - y_0)^2}}{(1-t)y_0 + ty_1}\,dt = |\log y_1 - \log y_0| = |\log(y_1/y_0)|.$$

For example, $d_H(yi, i) = |\log y|$ and the midpoint between $y_0 i$ and $y_1 i$ when $y_0 \neq y_1$ is $\sqrt{y_0 y_1}\, i$, which is (always) different from the Euclidean midpoint.

The action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathfrak{h}$ by linear fractional transformations preserves hyperbolic distances: for each $A \in \mathrm{SL}_2(\mathbf{R})$, $d_H(A(P), A(Q)) = d_H(P,Q)$ for all P and Q in $\mathfrak{h}$. A function $\mathfrak{h} \to \mathfrak{h}$ that preserves distances is called an isometry, and $\mathrm{SL}_2(\mathbf{R})$ acting by linear fractional transformation is the group of all orientation-preserving isometries of the hyperbolic plane.[11] An example of an isometry of $\mathfrak{h}$ that reverses orientation is $\tau \mapsto -\overline{\tau}$, or equivalently $x + yi \mapsto -x + yi$, and every orientation-reversing isometry is this example composed with the action by a matrix in $\mathrm{SL}_2(\mathbf{R})$.

Appendix B. A lattice sum

This section proves a result used in Section 4 to show Eisenstein series of weight $k \geq 4$ are absolutely convergent.

From calculus, the series $\sum_{n\geq 1} 1/n^s$ converges for $s > 1$ and diverges for $0 < s \leq 1$. We will generalize this result to a sum over the d-dimensional integral lattice $\mathbf{Z}^d$ for any $d \geq 1$. For any $\mathbf{x} = (x_1, \ldots, x_d)$ in $\mathbf{R}^d$, set $||\mathbf{x}|| = \sqrt{x_1^2 + \cdots + x_d^2}$. This is the length of $\mathbf{x}$.

Theorem B.1. *For $s > 0$, the infinite series* $\displaystyle\sum_{\mathbf{a}\in\mathbf{Z}^d-\{\mathbf{0}\}} \frac{1}{||\mathbf{a}||^s}$ *converges for $s > d$ and diverges for $0 < s \leq d$.*

Proof. We will first collect together all the terms of the same size (that is, all vectors in $\mathbf{Z}^d$ with the same length), and then use an identity called summation by parts, which is a discrete analogue of integration by parts. Then we will rewrite the desired sum as an integral, and our problem will be reduced to the fact that $\int_1^\infty dx/x^t$ converges for $t > 1$ and diverges for $0 < t \leq 1$.

The squared length $||\mathbf{a}||^2$ of any $\mathbf{a} \in \mathbf{Z}^d$ is a positive integer. For $n \geq 1$, set

$$r_d(n) = |\{\mathbf{a} \in \mathbf{Z}^d : ||\mathbf{a}||^2 = n\}|.$$

Then

$$\sum_{\mathbf{a}\in\mathbf{Z}^d-\{\mathbf{0}\}} \frac{1}{||\mathbf{a}||^s} = \sum_{\mathbf{a}\in\mathbf{Z}^d-\{\mathbf{0}\}} \frac{1}{(||\mathbf{a}||^2)^{s/2}} = \sum_{n\geq 1} \frac{r_d(n)}{n^{s/2}} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{r_d(n)}{n^{s/2}}.$$

For $n \geq 1$ set $S(n) = r_d(1) + \cdots + r_d(n) = |\{\mathbf{a} \in \mathbf{Z}^d : ||\mathbf{a}||^2 \leq n\}|$ and $S(0) = 0$, so $r_d(n) = S(n) - S(n-1)$ for $n \geq 1$. Then

$$\sum_{n=1}^{N} \frac{r_d(n)}{n^{s/2}} = \sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n^{s/2}}.$$

For a sum of the form $\sum_{n=1}^N u_n(v_n - v_{n-1})$, which resembles $\int u\,dv$, there is the following analogue of integration by parts, called summation by parts:

$$\sum_{n=1}^{N} u_n(v_n - v_{n-1}) = u_N v_N - u_1 v_0 - \sum_{n=1}^{N-1} v_n(u_{n+1} - u_n).$$

Using $u_n = 1/n^{s/2}$ and $v_n = S(n)$, so $v_0 = 0$, summation by parts implies

$$\sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n^{s/2}} = \frac{S(N)}{N^{s/2}} - \sum_{n=1}^{N-1} S(n)\left(\frac{1}{(n+1)^{s/2}} - \frac{1}{n^{s/2}}\right). \tag{B.1}$$

[11] Strictly speaking, since A and $-A$ act in the same way, the group of orientation-preserving isometries is $\mathrm{SL}_2(\mathbf{R})/\{\pm I_2\}$.

Write the difference $1/(n+1)^{s/2} - 1/n^{s/2}$ as an integral using the Fundamental Theorem of Calculus:

$$\frac{1}{(n+1)^{s/2}} - \frac{1}{n^{s/2}} = \int_n^{n+1} \frac{d}{dx}\left(\frac{1}{x^{s/2}}\right) dx = -\frac{s}{2}\int_n^{n+1} \frac{1}{x^{s/2+1}}\, dx.$$

Substituting this into (B.1),

$$\begin{aligned}\sum_{n=1}^{N} \frac{S(n)-S(n-1)}{n^{s/2}} &= \frac{S(N)}{N^{s/2}} + \frac{s}{2}\sum_{n=1}^{N-1} S(n)\int_n^{n+1} \frac{dx}{x^{s/2+1}} \\ &= \frac{S(N)}{N^{s/2}} + \frac{s}{2}\sum_{n=1}^{N-1} \int_n^{n+1} \frac{S(n)}{x^{s/2+1}}\, dx.\end{aligned}$$

For real $x \geq 1$, which need not be integers, set

$$S(x) = \sum_{1 \leq n \leq x} r_d(n) = |\{\mathbf{a} \in \mathbf{Z}^d : 1 \leq ||\mathbf{a}||^2 \leq x\}|,$$

so $S(x) = S(n)$ where $n \leq x < n+1$. Then

$$\begin{aligned}\sum_{n=1}^{N} \frac{S(n)-S(n-1)}{n^{s/2}} &= \frac{S(N)}{N^{s/2}} + \frac{s}{2}\sum_{n=1}^{N-1} \int_n^{n+1} \frac{S(x)}{x^{s/2+1}}\, dx \\ &= \frac{S(N)}{N^{s/2}} + \frac{s}{2}\int_1^{N} \frac{S(x)}{x^{s/2+1}}\, dx.\end{aligned}$$

To determine how $S(N)/N^{s/2}$ and the integral from 1 to N behave as $N \to \infty$, we will estimate $S(x)$ for large x using geometry.

The number $S(x)$ counts nonzero integral points inside the ball $\{\mathbf{x} \in \mathbf{R}^d : ||\mathbf{x}|| \leq \sqrt{x}\}$ with radius $\sqrt{x}$, and the number of such integral points is approximately the volume of that ball. A ball of radius r in $\mathbf{R}^d$ has volume $C_d r^d$ for some constant C_d depending only on d (for example, $C_2 = \pi$). Using $r = \sqrt{x}$, it turns out there are positive constants A_d and B_d such that

$$A_d x^{d/2} \leq S(x) \leq B_d x^{d/2} \tag{B.2}$$

for large x. Intuitively, (B.2) is due to volumes and lattice points counts of a ball in $\mathbf{R}^d$ growing at the same rate (for large radii). We give a more careful justification of (B.2) at the end.

Dividing through the inequality (B.2) by $x^{s/2+1}$, we get

$$\frac{A_d x^{(d-s)/2}}{x} \leq \frac{S(x)}{x^{s/2+1}} \leq \frac{B_d}{x^{(s-d)/2+1}} \tag{B.3}$$

If $0 < s \leq d$ then the first inequality in (B.3) tells us $S(x)/x^{s/2+1} \geq A_d/x$ for large x, which implies $\int_1^N S(x)/x^{s/2+1}\, dx \to \infty$ as $N \to \infty$, and thus our original lattice sum diverges.

If $s > d$ then the second inequality in (B.3) tells us $0 \leq S(x)/x^{s/2+1} \leq B_d/x^{1+\varepsilon}$ for large x, where $\varepsilon = (s-d)/2 > 0$, so $\int_1^N S(x)/x^{s/2+1}\, dx$ converges as $N \to \infty$. Using (B.2), $S(N)/N^{s/2} \leq B_d/N^{(s-d)/2} \to 0$, so our lattice sum converges and in fact

$$\sum_{\mathbf{a} \in \mathbf{Z}^d - \{\mathbf{0}\}} \frac{1}{||\mathbf{a}||^s} = \frac{s}{2}\int_1^{\infty} \frac{S(x)}{x^{s/2+1}}\, dx.$$

It remains to explain (B.2) for large x. Instead of counting integral vectors $\mathbf{a}$ that satisfy a Euclidean-norm condition $||\mathbf{a}|| \leq R$ for some $R > 0$, let's first count integral vectors $\mathbf{a}$ that are bounded for another norm on $\mathbf{R}^d$: $||\mathbf{a}||_{\max} \leq R$, where

$$||\mathbf{x}||_{\max} = \max(|x_1|, \ldots, |x_d|).$$

Set

$$S_{\max}(R) := |\{\mathbf{a} \in \mathbf{Z}^d : ||\mathbf{a}||_{\max} \leq R\}|.$$

The number of integers n satisfying $-R \leq n \leq R$ is $2\lfloor R \rfloor + 1$ (check this when $R = 1$), so $S_{\max}(R) = (2\lfloor R \rfloor + 1)^d$ from the way $|| \cdot ||_{\max}$ is defined. When $R \geq 1$, $R \leq 2\lfloor R \rfloor + 1 \leq 3R$, so

$$R^d \leq S_{\max}(R) \leq 3^d R^d. \tag{B.4}$$

Qualitatively, this is the type of upper and lower bound we want for $S(x)$ in (B.2), with $x = R^2$, so let's bound $|| \cdot ||$ in terms of $|| \cdot ||_{\max}$ from above and below in order to convert (B.4) into (B.2).

Check that $||\mathbf{x}||_{\max} \leq ||\mathbf{x}|| \leq \sqrt{d}||\mathbf{x}||_{\max}$ for all $\mathbf{x}$ in $\mathbf{R}^d$, so

$$||\mathbf{x}|| \leq R \Longrightarrow ||\mathbf{x}||_{\max} \leq R \quad \text{and} \quad ||\mathbf{x}||_{\max} \leq R \Longrightarrow ||\mathbf{x}|| \leq \sqrt{d}R.$$

Therefore when $x \geq 1$, $S(x) = |\{\mathbf{a} \in \mathbf{Z}^d : ||\mathbf{a}|| \leq \sqrt{x}\}| \leq S_{\max}(\sqrt{x}) \leq 3^d x^{d/2}$ by (B.4) since $\sqrt{x} \geq 1$. To get a lower bound on $S(x)$, if $||\mathbf{a}||_{\max} \leq R$ then $||\mathbf{a}|| \leq \sqrt{d}R = \sqrt{dR^2}$, so $S_{\max}(R) \leq S(dR^2)$. Thus $S(x) \geq S_{\max}(\sqrt{x/d})$, and if $x \geq d$ (so $\sqrt{x/d} \geq 1$) we get $S_{\max}(\sqrt{x/d}) \geq \sqrt{x/d}^{\,d} = x^{d/2}/d^{d/2}$ by (B.4). We have proved (B.2) when $x \geq d$ using $A_d = 1/d^{d/2}$ and $B_d = 3^d$. $\square$