Mock and quantum modular forms

Amanda Folsom (Amherst College)

Ramanujan's mock theta functions

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1887 - 1920

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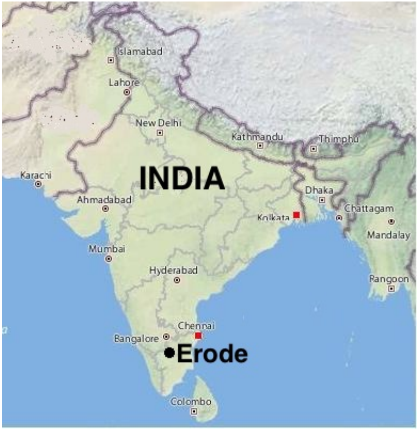
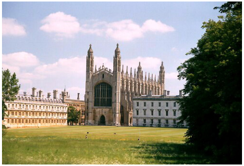


History

S. Ramanujan

- Encountered math at a young age
- Ramanujan failed out of school
- Worked as a shipping clerk, pursued mathematics on his own

History



- Ramanujan wrote letters to mathematics professors in Cambridge, England
- Initially, all were ignored

History

- G.H. Hardy recognized Ramanujan's talent
- Hardy invited Ramanujan to Cambridge
- Ramanujan produced nearly 4000 original and deep results

G.H. Hardy

Ramanujan's last letter

Dear Hardy,

"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "Mock" ϑ-functions...I am sending you with this letter some examples."

- **S. Ramanujan, January 12, 1920**

Ramanujan's mock theta functions

Examples.

$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

Ramanujan's mock theta functions

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$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

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Def. $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j),\ n \in \mathbb{N}, \qquad (a;q)_0 := 1.$

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$$\omega(q) := \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q;q^2)_n^2} = 1 + \frac{q^4}{(1-q)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2} + \cdots$$

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$$F_2(q) := \sum_{n\geq 0} \frac{q^{n(n+1)}}{(q^{n+1};q)_{n+1}} = \frac{1}{(1-q)} + \frac{q^2}{(1-q^2)(1-q^3)} + \cdots$$

$\vdots$

Def. $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \; n \in \mathbb{N}, \qquad (a;q)_0 := 1.$

Ramanujan's mock theta functions

Atkin, Andrews, Dyson, Hardy, Ramanujan, Selberg, Swinnerton-Dyer, Watson, etc. studied

- asymptotic behaviors
- analytic properties
- combinatorial properties
- q-hypergeometric identites

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- **combinatorial properties**
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Partition numbers

Definition

A **partition** of a natural number n is any way to write n as a non-increasing sum of natural numbers.

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The **partition function** $p(n) :=$ number of partitions of n.

Integer partitions

$1 = 1$ $\qquad$ $p(1) = 1$

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$2 = 2, \; 1 + 1$ $\quad$ $p(2) = 2$

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$2 = 2, \ 1 + 1$ $\quad$ $p(2) = 2$

$3 = 3, \ 2 + 1, \ 1 + 1 + 1$ $\quad$ $p(3) = 3$

Integer partitions

$$1 = 1 \qquad \qquad p(1) = 1$$

$$2 = 2,\ 1+1 \qquad \qquad p(2) = 2$$

$$3 = 3,\ 2+1,\ 1+1+1 \qquad \qquad p(3) = 3$$

$$4 = 4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1 \qquad \qquad p(4) = 5$$

$$\vdots \qquad \qquad \vdots$$

Integer partitions: 1700s - present


Euler

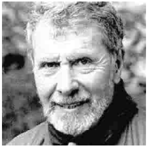

Hardy

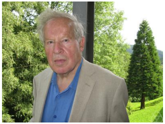

Watson

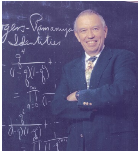

Ramanujan


Rademacher


Dyson


Atkin


Swinnerton-Dyer


Andrews


Ono

The partition function $p(n)$

The partition function $p(n)$

1

The partition function $p(n)$

1, 2

The partition function $p(n)$

1, 2, 3

The partition function $p(n)$

1, 2, 3, 5

The partition function $p(n)$

1, 2, 3, 5, 7

The partition function $p(n)$

1, 2, 3, 5, 7, 11

The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15

The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22

The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30

The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42

The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56

The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77

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1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101

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1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135

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1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385

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$\nwarrow p(115)$

Generating functions

The generating function $\mathcal{A}(q)$ for a sequence $\{a(n)\}_{n\geq 0}$ is

$$\mathcal{A}(q) := \sum_{n\geq 0} a(n)q^n = a(0) + a(1)q + a(2)q^2 + a(3)q^3 + \cdots$$

The partition generating function

The partition generating function

$$\mathcal{P}(q) := \sum_{n\geq 0} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots$$

The partition generating function

Theorem (Euler, 1700s)

For $|q| < 1$,

$$\mathcal{P}(q) := \sum_{n\geq 0} p(n)q^n = \prod_{m\geq 1} \frac{1}{1-q^m}.$$

The partition generating function

Theorem (Euler, 1700s)

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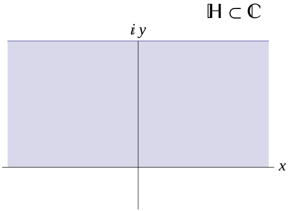
Geometric Series: $\sum_{j\geq 0} q^j = \frac{1}{1-q}$

Partitions and modular forms

A key to answering certain questions about $\mathcal{P}(q)$, hence $p(n)$, is its relationship to modular forms.

Modular forms

Modular forms $g : \mathbb{H} := \{\tau \in \mathbb{C} \ |\text{Im}(\tau) > 0\} \longrightarrow \mathbb{C}$ satisfy:



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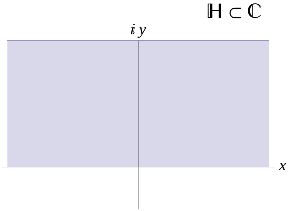

$$g(\gamma \cdot \tau) = \boxed{} \cdot g(\tau)$$

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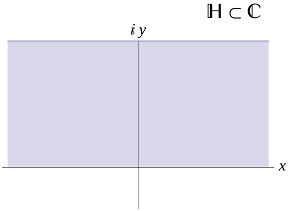

$$g(\gamma \cdot \tau) = \boxed{\qquad\qquad} \cdot g(\tau)$$

$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) = \{2 \times 2 \;\; \text{integer matrices, determinant} \;\; 1\}$

$\gamma \cdot \tau := \frac{a\tau+b}{c\tau+d} \in \mathbb{H}$

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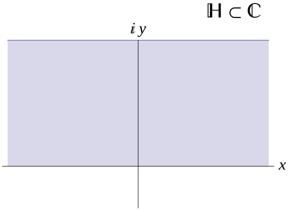

$$g(\gamma \cdot \tau) = \epsilon(\gamma)(c\tau + d)^k \cdot g(\tau)$$

$k \in \frac{1}{2}\mathbb{Z}$

$|\epsilon(\gamma)| = 1$

Modular forms

Definition (Dedekind's η-function)

Let $q = q_\tau := e^{2\pi i\tau}$, where $\tau \in \mathbb{H}$.

$$\eta(\tau) := q^{1/24} \prod_{n\geq 1} (1 - q^n).$$

Modular forms

Proposition

The η-function is a modular form of weight $1/2$. In particular,

$$\eta\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\tau\right)=\eta(\tau+1)=\zeta_{24}\cdot\eta(\tau),$$

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Notation: $\zeta_N := e^{\frac{2\pi i}{N}}$ is an Nth root of unity.

Modular forms

To understand aspects of the combinatorial function $p(n)$, we can exploit the modularity of the weight $\frac{1}{2}$ form

$$\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n) \overset{Euler}{=} q^{1/24} \left(\sum_{n \geq 0} p(n) q^n \right)^{-1}$$

The partition function

Remark. A combinatorial proof shows that we may also rewrite

$$\mathcal{P}(q) = \sum_{n \geq 0} p(n) q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n^2}.$$

q-hypergeometric series

Remark. Series of this “shape” are called q-hypergeometric series.

q-hypergeometric series

Therefore:

$$q^{1/24}\eta^{-1}(\tau) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2}$$

q-hypergeometric series

Therefore:

$$q^{1/24}\eta^{-1}(\tau) = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2}$$

Recall: Ramanujan's mock theta function

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

Partition ranks

Definition (Dyson)

Let λ be a partition. Then

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda.$$

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Example: the partition $\lambda = 3 + 3 + 3 + 1$ has

$$\text{rank}(\lambda) = 3 - 4 = -1.$$

Partition ranks

Lemma

The mock theta function f satisfies

$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = \sum_{n \geq 0} (p_e(n) - p_o(n))\, q^n$$

where $p_e(n) := p(n \mid \text{even rank}),\ \ p_o(n) := p(n \mid \text{odd rank}).$

Combinatorial modular(?) q-hypergeometric series

$$q^{1/24}\eta^{-1}(\tau) = \sum_{n\geq 0} p(n)q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2}$$

modular✓ combinatorial✓ q-hypergeometric✓

$$f(q) = \sum_{n\geq 0}(p_e(n) - p_0(n))q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

modular??? combinatorial✓ q-hypergeometric✓

Modular forms and mock theta functions

Question

What roles do mock theta functions play within the theory of modular forms?

Modular forms and mock theta functions

Let $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, and $q = e^{2\pi i \tau}$.

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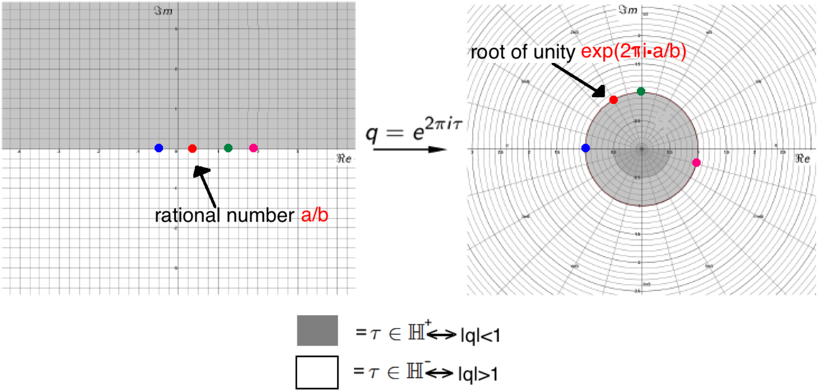
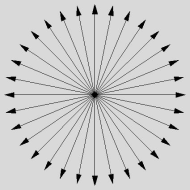


Modular forms and mock theta functions

Ramanujan's last letter (1920).

- Asymptotics, near roots of unity, of "Eulerian" (i.e. q-hypergeometric) series.



Modular forms and mock theta functions

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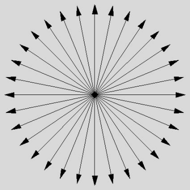

i.e. q-series "similar" in shape to

$$f(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

Modular forms and mock theta functions

Question (Ramanujan)

Must Eulerian series with "similar asymptotics" be the sum of a modular form and a function which is $O(1)$ (i.e. bounded) at all roots of unity?

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Ramanujan's Answer

"The answer is it is not necessarily so....I have not proved rigorously that it is not so. But I have constructed a number of examples..."

Modular forms and mock theta functions

Define the modular form $b(q)$ by

$$b(q) := \prod_{m \geq 1} (1 - q^{2m-1}) \times \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}$$

Modular forms and mock theta functions

Define the modular form $b(q)$ by

$$b(q) := \prod_{m\geq 1}(1 - q^{2m-1}) \times \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$$

$$= q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)} \qquad (q = e^{2\pi i \tau})$$

Modular forms and mock theta functions

Claim (Ramanujan)

If q radially approaches an even order $2k$ root of unity, then

$$f(q) - (-1)^k b(q) = O(1).$$

F. Dyson (1987)

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"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered.

...Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi.

...This remains a challenge for the future..."

Modular forms and mock theta functions

Theorem (Zwegers, '02)

Ramanujan's mock theta functions may be "completed" to obtain vector valued, non-holomorphic modular forms.

A completion

Example:

$$F(\tau) := (q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}), 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T$$

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$$F(\tau) := (q^{-\frac{1}{24}}f(q), 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}), 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T$$

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

$$\omega(q) := \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q;q^2)_n^2}$$

A completion

Example (cont.):

$$G(\tau) := 2i\sqrt{3}\left(\int_{-\overline{\tau}}^{i\infty} \frac{g_1(z)\ dz}{\sqrt{-i(\tau+z)}}, \int_{-\overline{\tau}}^{i\infty} \frac{g_0(z)\ dz}{\sqrt{-i(\tau+z)}}, \int_{-\overline{\tau}}^{i\infty} \frac{-g_2(z)\ dz}{\sqrt{-i(\tau+z)}}\right)^T$$

A completion

Example (cont.):

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$$g_j(z) = \text{weight } \tfrac{3}{2} \text{ modular } \Theta\text{−functions}$$

Some results of Zwegers

Theorem (Zwegers)

The function

$$H(\tau) := F(\tau) - G(\tau)$$

is a vector-valued weight $\frac{1}{2}$ *non-holomorphic modular form. In particular,* $H(\tau)$ *transforms as*

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau),$$

$$H(-1/\tau) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$$

Harmonic Maass forms

Definition (Bruinier-Funke, '04)

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A *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(4N)$ is a smooth $M : \mathbb{H} \to \mathbb{C}$ satisfying

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(1) transformation law: $\forall A \in \Gamma_0(4N)$, $\tau \in \mathbb{H}$,

$$M(A\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (c\tau + d)^k M(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ (c\tau + d)^k M(\tau), & k \in \mathbb{Z} \end{cases}$$

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(2) harmonic: $\Delta_k M = 0$, where
$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$
$(\tau = x + iy)$

Harmonic Maass forms

Definition (Bruinier-Funke, '04)

A *harmonic Maass form* of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(4N)$ is a smooth $M : \mathbb{H} \to \mathbb{C}$ satisfying

(1) transformation law: $\forall A \in \Gamma_0(4N)$, $\tau \in \mathbb{H}$,

$$M(A\tau) = \begin{cases} \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (c\tau + d)^k M(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ (c\tau + d)^k M(\tau), & k \in \mathbb{Z} \end{cases}$$

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(3) M satisfies a suitable growth condition in the cusps.

Harmonic Maass forms

Technical remarks.

1. $\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},$

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5. We take the principal branch of the holomorphic $\sqrt{\cdot}$

Harmonic Maass forms

Condition (3): $\exists \ P_M \in \mathbb{C}[q^{-1}]$ such that

$$M(\tau) - P_M(\tau) = O(e^{-\epsilon y}),$$

for some $\epsilon > 0$ as $y \to \infty$.

Harmonic Maass forms

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$$\Gamma(a,x) := \int_x^\infty t^{a-1} e^{-t} dt$$

Mock theta functions and mock modular forms

Fact: Ramanujan's mock theta functions are examples of "holomorphic parts" M^+ of harmonic Maass forms.

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Ex. $$M^+(\tau) = q^{-\frac{1}{24}} \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

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Fact: Ramanujan's mock theta functions are examples of "holomorphic parts" M^+ of harmonic Maass forms.

Ex. $$M^+(\tau) = q^{-\frac{1}{24}} \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

Definition (Zagier). **A mock modular form** := a holomorphic part of a HMF.

Applications

- q-series
- modular L-functions
- combinatorics
- generalized Borcherds products
- Moonshine
- Donaldson invariants
- mathematical physics
- quantum modular forms

$\vdots$

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Let $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$, and $q = e^{2\pi i \tau}$.

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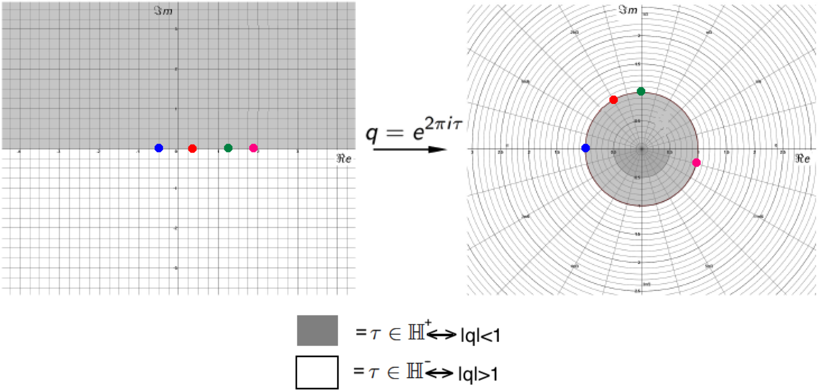

Quantum modular forms

Let $g : \mathbb{H} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}), \quad \tau \in \mathbb{H}.$

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Modular transformation:

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(\gamma)(c\tau + d)^k g(\tau)$$

or rather,

$$g(\tau) - \epsilon(\gamma)^{-1}(c\tau + d)^{-k} g\left(\frac{a\tau + b}{c\tau + d}\right) = 0$$

Quantum modular forms

Let $g : \mathbb{Q} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}), \quad x \in \mathbb{Q}.$

Modular transformation:

$$g(x) - \epsilon(\gamma)^{-1}(cx+d)^{-k} g\left(\frac{ax+b}{cx+d}\right) = ?$$

Quantum modular forms

Definition (Zagier '10)

A **quantum modular form of weight** k ($k \in \frac{1}{2}\mathbb{Z}$) is function $g : \mathbb{Q} \setminus S \to \mathbb{C}$, for some discrete subset S, such that

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$$h_\gamma(x) = h_{g,\gamma}(x) := g(x) - \epsilon(\gamma)^{-1}(cx+d)^{-k} g\left(\frac{ax+b}{cx+d}\right)$$

satisfy a suitable property of continuity or analyticity in $\mathbb{R}$.

Quantum modular forms

Example. Kontsevich's "strange" function ($x \in \mathbb{Q}$):

$$\phi(x) := e^{\frac{\pi i x}{12}} \sum_{n=0}^{\infty} \left(e^{2\pi i x}; e^{2\pi i x}\right)_n$$

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Exercise: ϕ converges for any rational number x.

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The function ϕ is a quantum modular form of weight $3/2$,

Quantum modular forms

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The function ϕ is a quantum modular form of weight 3/2, i.e.

$$\phi(x+1) = \zeta_{24}\phi(x), \quad \phi(x) \mp \zeta_8 |x|^{-\frac{3}{2}} \phi(-1/x) = h(x),$$

where h is a real analytic function (except at 0).

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Quantum modular forms

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$$U(\tau) := q^{-\frac{1}{24}} \sum_{n=0}^{\infty} (q;q)_n^2 q^{n+1} = q^{-\frac{1}{24}} \sum_{\substack{m \in \mathbb{Z} \\ n \geq 1}} u(m,n)(-1)^m q^n.$$

Quantum modular forms

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Let $x \in \mathbb{Q}$. We have that

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Modular forms and mock theta functions

Ramanujan's claim revisited (F - Ono - Rhoades)

Modular forms and mock theta functions

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If q radially approaches an even order $2k$ root of unity, then

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$$b(q) := \prod_{m\geq 1}(1 - q^{2m-1}) \times \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)}$$

$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

Numerics

As $q \to -1$, we computed

$$f(-0.994) \sim -1 \cdot 10^{31},\ f(-0.996) \sim -1 \cdot 10^{46},\ f(-0.998) \sim -6 \cdot 10^{90} \ldots$$

Numerics (cont.)

Ramanujan's claim gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\ldots$	$3.969\ldots$	$3.976\ldots$	$3.984\ldots$	$3.992\ldots$

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This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

Numerics (cont.)

q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2 \cdot 10^6 - 4.6 \cdot 10^6 i$	$2 \cdot 10^8 - 4 \cdot 10^8 i$	$1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

Numerics (cont.)

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This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$

Questions

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What are the $O(1)$ constants in

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = O(1)?$$

How do they arise?

Ramanujan's radial limits

Theorem (F-Ono-Rhoades)

If ζ is an even $2k$ order root of unity, then

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

Radial limits

Remark. We prove this as a special case of a more general theorem involving:

Radial limits

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$$R(w;q) := \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} \mathcal{N}(m,n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n},$$

mock modular rank function [Bringmann-Ono]

$$C(w;q) := \sum_{n\geq 0}\sum_{m\in\mathbb{Z}} \mathcal{M}(m,n) w^m q^n = \frac{(q;q)_\infty}{(wq;q)_\infty (w^{-1}q;q)_\infty},$$

modular crank function

$$U(w;q) := \sum_{n\geq 1}\sum_{m\in\mathbb{Z}} u(m,n)(-w)^m q^n = \sum_{n=0}^{\infty} (wq;q)_n (w^{-1}q;q)_n q^{n+1}.$$

quantum modular unimodal function [B-O-P-R]

Combinatorial "modular" forms

Here,

$$\mathcal{N}(m,n) := \#\{\text{partitions } \lambda \text{ of } n \mid \text{rank}(\lambda) = m\},$$

$$\mathcal{M}(m,n) := \#\{\text{partitions } \lambda \text{ of } n \mid \text{crank}(\lambda) = m\},$$

$$u(m,n) := \#\{\text{size } n \text{ strongly unimodal sequences with rank } m\}.$$

Radial limits

Theorem (F-Ono-Rhoades)

If $\zeta_b = e^{\frac{2\pi i}{b}}$ and $1 \leq a < b$, then for every suitable root of unity ζ there is an explicit integer c for which

$$\lim_{q \to \zeta} \left(R(\zeta_b^a; q) - \zeta_{b^2}^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).$$

Radial limits

Remark

The first theorem is the special case $a = 1$, $b = 2$, using the facts that

$$R(-1; q) = f(q) \quad \text{and} \quad C(-1; q) = b(q).$$

Radial limits

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Remark

With the specialization $(w; q) \mapsto (\zeta_b^a; \zeta)$, the function $U(\zeta_b^a; \zeta)$ is a finite sum.

Radial limits

Theorems $\implies$

$$\lim_{q\to\zeta} (\text{Mock } \vartheta - \text{Mod. Form}) = \text{Quantum MF}.$$

Radial limits

Theorems $\Longrightarrow$

$$\lim_{q \to \zeta} (\text{Mock } \vartheta - \text{Mod. Form}) = \text{Quantum MF.}$$

$$\lim_{q \to \zeta} (\text{rank function} - \text{crank function}) = \text{unimodal function.}$$

Upper and lower half-planes

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Proof philosophy.

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Example

For Ramanujan's $f(q)$, remarkably we have

$$f(q^{-1}) = \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n^2} = 1 + q - q^2 + 2q^3 - 4q^4 + \dots$$

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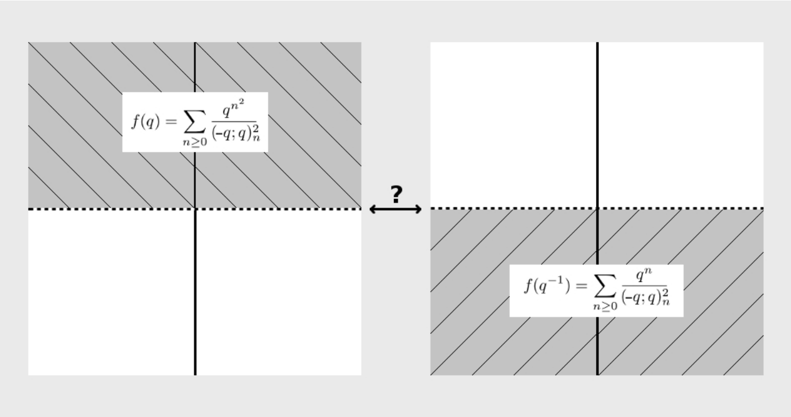
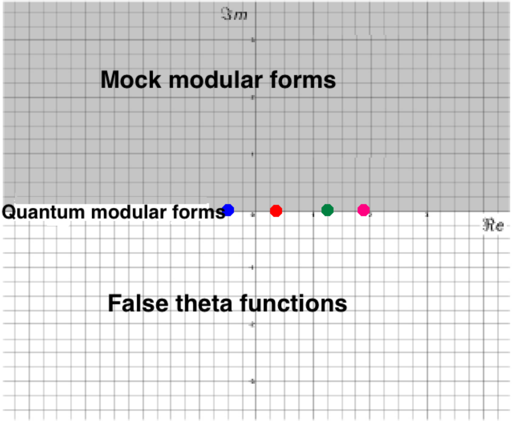
Remark. Under $\tau \leftrightarrow q = e^{2\pi i\tau}$, $f(q)$ is defined on both $\mathbb{H} \cup \mathbb{H}^{-}$.

Upper and lower half-planes



Upper and lower half-planes

A larger framework:



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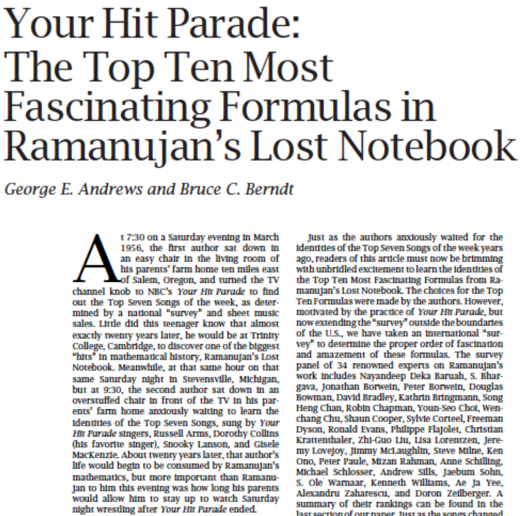

Your Hit Parade: The Top Ten Most Fascinating Formulas in Ramanujan's Lost Notebook

George E. Andrews and Bruce C. Berndt

At 7:30 on a Saturday evening in March 1956, the first author sat down in an easy chair in the living room of his parents' farm home ten miles east of Salem, Oregon, and turned the TV channel knob to NBC's *Your Hit Parade* to find out the Top Seven Songs of the week, as determined by a national "survey" and sheet music sales. Little did this teenager know that almost exactly twenty years later, he would be at Trinity College, Cambridge, to discover one of the biggest "hits" in mathematical history, Ramanujan's Lost Notebook. Meanwhile, at that same hour on that same Saturday night in Stevensville, Michigan, but at 9:30, the second author sat down in an overstuffed chair in front of the TV in his parents' farm home anxiously waiting to learn the identities of the Top Seven Songs, sung by *Your Hit Parade* singers, Russell Arms, Dorothy Collins (his favorite singer), Snooky Lanson, and Gisele MacKenzie. About twenty years later, that author's life would begin to be consumed by Ramanujan's mathematics, but more important than Ramanujan to him this evening was how long his parents would allow him to stay up to watch Saturday night wrestling after *Your Hit Parade* ended.

Just as the authors anxiously waited for the identities of the Top Seven Songs of the week years ago, readers of this article must now be brimming with unbridled excitement to learn the identities of the Top Ten Most Fascinating Formulas from Ramanujan's Lost Notebook. The choices for the Top Ten Formulas were made by the authors. However, motivated by the practice of *Your Hit Parade*, but now extending the "survey" outside the boundaries of the U.S., we have taken an international "survey" to determine the proper order of fascination and amazement of these formulas. The survey panel of 34 renowned experts on Ramanujan's work includes Nayandeep Deka Baruah, S. Bhargava, Jonathan Borwein, Peter Borwein, Douglas Bowman, David Bradley, Kathrin Bringmann, Song Heng Chan, Robin Chapman, Youn-Seo Choi, Wenchang Chu, Shaun Cooper, Sylvie Corteel, Freeman Dyson, Ronald Evans, Philippe Flajolet, Christian Krattenthaler, Zhi-Guo Liu, Lisa Lorentzen, Jeremy Lovejoy, Jimmy McLaughlin, Steve Milne, Ken Ono, Peter Paule, Mizan Rahman, Anne Schilling, Michael Schlosser, Andrew Sills, Jaebum Sohn, S. Ole Warnaar, Kenneth Williams, Ae Ja Yee, Alexandru Zaharescu, and Doron Zeilberger. A summary of their rankings can be found in the last section of our paper. Just as the songs changed

Radial limits

Poll to rank most fascinating formulas in the “lost” notebook:

1. **Dyson’s Ranks**.
2. **Mock ϑ-functions**.
3. **Andrews-Garvan Crank.**
4. Continued fraction with three limit points.
5. **Early QMFs**: “Sums of Tails” of Euler’s Products.

Related work:

- Bajpai-Kimport-Liang-Ma-Ricci
- Bringmann-Creutzig-Rolen
- Bringmann-F-Rhoades
- Bringmann-Rolen
- Bryson-Ono-Pittman-Rhoades
- F-Ki-Truong Vu-Yang
- F-Ono-Rhoades
- Hikami-Lovejoy
- Joo-Löbrich
- Rolen-Schneider
- Zagier
- etc.

Thank you