Mock and quantum modular forms

Ramanujan’s mock theta functions
Mock and quantum modular forms

Ramanujan’s mock theta functions

1887 - 1920
Ramanujan’s mock theta functions

1887 - 1920
History

S. Ramanujan

- Encountered math at a young age
- Ramanujan failed out of school
- Worked as a shipping clerk, pursued mathematics on his own
Ramanujan wrote letters to mathematics professors in Cambridge, England

Initially, all were ignored
History

- G.H. Hardy recognized Ramanujan’s talent
- Hardy invited Ramanujan to Cambridge
- Ramanujan produced nearly 4000 original and deep results
Dear Hardy,

“I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call “Mock” \( \vartheta \)-functions...I am sending you with this letter some examples.”

- S. Ramanujan, January 12, 1920
Ramanujan’s mock theta functions

Examples.

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} \]
Examples.

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]
Ramanujan’s mock theta functions

Examples.

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)^2_n} = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]

Def. \( (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad n \in \mathbb{N}, \quad (a; q)_0 := 1. \)
Ramanujan’s mock theta functions

Examples.

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]

\[ \omega(q) := \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(q; q^2)_n^2} = 1 + \frac{q^4}{(1 - q)^2} + \frac{q^{12}}{(1 - q)^2(1 - q^3)^2} + \cdots \]

**Def.** \((a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \ n \in \mathbb{N}, \ (a; q)_0 := 1.\)
Ramanujan’s mock theta functions

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\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots \]

\[ \omega(q) := \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(q; q^2)_n^2} = 1 + \frac{q^4}{(1 - q)^2} + \frac{q^{12}}{(1 - q)^2(1 - q^3)^2} + \cdots \]

\[ F_2(q) := \sum_{n \geq 0} \frac{q^{n(n+1)}}{(q^{n+1}; q)_{n+1}} = \frac{1}{(1 - q)} + \frac{q^2}{(1 - q^2)(1 - q^3)} + \cdots \]

\[ \vdots \]

**Def.** \((a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad n \in \mathbb{N}, \quad (a; q)_0 := 1.\]
Ramanujan’s mock theta functions

Atkin, Andrews, Dyson, Hardy, Ramanujan, Selberg, Swinnerton-Dyer, Watson, etc. studied

- asymptotic behaviors
- analytic properties
- combinatorial properties
- $q$-hypergeometric identities
Ramanujan’s mock theta functions

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- asymptotic behaviors
- analytic properties
- **combinatorial properties**
- \(q\)-hypergeometric identities
Partition numbers

Definition

A **partition** of a natural number $n$ is any way to write $n$ as a non-increasing sum of natural numbers.
Partition numbers

**Definition**

A *partition* of a natural number $n$ is any way to write $n$ as a non-increasing sum of natural numbers.

The *partition function* $p(n) :=$ number of partitions of $n$. 
Integer partitions

1 = 1

\[ p(1) = 1 \]
Integer partitions

\begin{align*}
1 &= 1 \\
2 &= 2, \ 1 + 1 \\
p(1) &= 1 \\
p(2) &= 2
\end{align*}
Integer partitions

1 = 1  \quad p(1) = 1

2 = 2, \ 1 + 1  \quad p(2) = 2

3 = 3, \ 2 + 1, \ 1 + 1 + 1  \quad p(3) = 3
Integer partitions

\[ 1 = 1 \quad p(1) = 1 \]

\[ 2 = 2, \ 1 + 1 \quad p(2) = 2 \]

\[ 3 = 3, \ 2 + 1, \ 1 + 1 + 1 \quad p(3) = 3 \]

\[ 4 = 4, \ 3 + 1, \ 2 + 2, \ 2 + 1 + 1, \ 1 + 1 + 1 + 1 \quad p(4) = 5 \]

\[ : \quad : \]

Mock and quantum modular forms
Mock and quantum modular forms

Integer partitions: 1700s - present

Euler, Hardy, Watson, Ramanujan, Rademacher, Dyson, Atkin, Swinnerton-Dyer, Andrews, Ono
The partition function $p(n)$
The partition function $p(n)$

$$p(115)$$
The partition function $p(n)$

1, 2
The partition function $p(n)$

1, 2, 3
Mock and quantum modular forms

The partition function \( p(n) \)

\[ 1, 2, 3, 5 \]
The partition function $p(n)$

$1, 2, 3, 5, 7$
The partition function $p(n)$

1, 2, 3, 5, 7, 11
Mock and quantum modular forms

The partition function $p(n)$

$1, 2, 3, 5, 7, 11, 15$
The partition function $p(n)$

$1, 2, 3, 5, 7, 11, 15, 22$
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42
The partition function \( p(n) \)
1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176
The partition function \( p(n) \)

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297
The partition function $p(n)$

$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385$
The partition function $p(n)$

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226, 239943, 281589, 329931, 386155, 451276, 526823, 614154, 715220, 831820, 966467, 1121505, 1300156, 1505499, 1741630, 2012558, 2323520, 2679689, 3087735, 3554345, 4087968, 4697205, 5392783, 6185689, 7089500, 8118264, 9289091, 10619863, 12132164, 13848650, 15796476, 18004327, 20506255, 23338469, 26543660, 30167357, 34262962, 38887673, 44108109, 49995925, 56634173, 64112359, 72533807, 82010177, 92669720, 104651419, 118114304, 133230930, 150198136, 169229875, 190569292, 214481126, 241265379, 271248950, 304801365, 342325709, 384276336, 431149389, 483502844, 541946240, 607163746, 679903203, 761002156, 851376628, 952050665, 1064144451
The partition function \( p(n) \)

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 26015, 31185, 37338, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226, 239943, 281589, 329931, 386155, 451276, 526823, 614154, 715220, 831820, 966467, 1121505, 1300156, 1505499, 1741630, 2012558, 2323520, 2679689, 3087735, 3554345, 4087968, 4697205, 5392783, 6185689, 7089500, 8118264, 9289091, 10619863, 12132164, 13848650, 15796476, 18004327, 20506255, 23338469, 26543660, 30167357, 34262962, 38887673, 44108109, 49995925, 56634173, 64112359, 72533807, 82010177, 92669720, 104651419, 118114304, 133230930, 150198136, 169229875, 190569292, 214481126, 241265379, 271248950, 304801365, 342325709, 384276336, 431149389, 483502844, 541946240, 607163746, 679903203, 761002156, 851376628, 952050665, 1064144451 

\( p(115) \)
Generating functions

The generating function $A(q)$ for a sequence $\{a(n)\}_{n \geq 0}$ is

$$A(q) := \sum_{n \geq 0} a(n)q^n = a(0) + a(1)q + a(2)q^2 + a(3)q^3 + \cdots$$
The partition generating function

\[ \mathcal{P}(q) := \sum_{n \geq 0} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots \]
The partition generating function

Theorem (Euler, 1700s)

For $|q| < 1$,

$$P(q) := \sum_{n \geq 0} p(n)q^n = \prod_{m \geq 1} \frac{1}{1 - q^m}.$$
The partition generating function

Theorem (Euler, 1700s)

For $|q| < 1$,

$$\mathcal{P}(q) := \sum_{n \geq 0} p(n)q^n = \prod_{m \geq 1} \frac{1}{1 - q^m}.$$ 

Geometric Series: $$\sum_{j \geq 0} q^j = \frac{1}{1 - q}$$
Mock and quantum modular forms

Partitions and modular forms

A key to answering certain questions about $\mathcal{P}(q)$, hence $p(n)$, is its relationship to modular forms.
Mock and quantum modular forms

Modular forms

Modular forms $g : \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \rightarrow \mathbb{C}$ satisfy:

$$g(\gamma \cdot \tau) = \gamma \cdot g(\tau)$$

for $\gamma \in \text{SL}_2(\mathbb{Z})$. 

Here is a diagram of the upper half-plane $\mathbb{H} \subset \mathbb{C}$.
Mock and quantum modular forms

Modular forms

Modular forms $g : \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \rightarrow \mathbb{C}$ satisfy:

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Mock and quantum modular forms

**Modular forms** $g : \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \to \mathbb{C}$ satisfy:

$$g(\gamma \cdot \tau) = \frac{a\tau + b}{c\tau + d} \in \mathbb{H}$$
**Modular forms**

**Modular forms** $g : \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \rightarrow \mathbb{C}$ satisfy:

$$g(\gamma \cdot \tau) = \epsilon(\gamma)(c\tau + d)^k \cdot g(\tau)$$

where $k \in \frac{1}{2} \mathbb{Z}$ and $|\epsilon(\gamma)| = 1$. 

**Diagram:**

- $\mathbb{H} \subset \mathbb{C}$
- $i_y$
- $x$

---

Mock and quantum modular forms
Definition (Dedekind’s $\eta$-function)

Let $q = q_\tau := e^{2\pi i \tau}$, where $\tau \in \mathbb{H}$.

$$\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n).$$
Proposition

The $\eta$-function is a modular form of weight 1/2. In particular,

$$\eta \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tau \right) = \eta(\tau + 1) = \zeta_{24} \cdot \eta(\tau),$$

$$\eta \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau \right) = \eta(-1/\tau) = \sqrt{-i\tau} \cdot \eta(\tau).$$
Proposition

The \( \eta \)-function is a modular form of weight \( 1/2 \). In particular,

\[
\eta \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tau \right) = \eta (\tau + 1) = \zeta_{24} \cdot \eta (\tau),
\]

\[
\eta \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau \right) = \eta (-1/\tau) = \sqrt{-i\tau} \cdot \eta (\tau).
\]

Notation: \( \zeta_N := e^{\frac{2\pi i}{N}} \) is an \( N \)th root of unity.
To understand aspects of the combinatorial function $p(n)$, we can exploit the modularity of the weight $\frac{1}{2}$ form

$$\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n)^{Euler} q^{1/24} \left( \sum_{n \geq 0} p(n) q^n \right)^{-1}$$
Remark. A combinatorial proof shows that we may also rewrite

\[ \mathcal{P}(q) = \sum_{n \geq 0} p(n)q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n^2}. \]
q-hypergeometric series

**Remark.** Series of this “shape” are called $q$-hypergeometric series.
Therefore:

\[ q^{1/24} \eta^{-1}(\tau) = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n^2} \]
Therefore:

\[ q^{1/24} \eta^{-1}(\tau) = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n^2} \]

Recall: Ramanujan’s mock theta function

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} \]
Partition ranks

Definition (Dyson)

Let \( \lambda \) be a partition. Then

\[
\text{rank}(\lambda) \coloneqq \text{largest part of } \lambda - \text{number of parts of } \lambda.
\]
Mock and quantum modular forms

Partition ranks

Definition (Dyson)
Let $\lambda$ be a partition. Then

$$\text{rank}(\lambda) := \text{largest part of } \lambda - \text{number of parts of } \lambda.$$  

Example: the partition $\lambda = 3 + 3 + 3 + 1$ has
$$\text{rank}(\lambda) = 3 - 4 = -1.$$
Lemma

The mock theta function $f$ satisfies

$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} = \sum_{n \geq 0} (p_e(n) - p_o(n)) q^n$$

where $p_e(n) := p(n \mid \text{even rank})$, $p_o(n) := p(n \mid \text{odd rank})$. 

Combinatorial modular (?) $q$-hypergeometric series

\[
q^{1/24}\eta^{-1}(\tau) = \sum_{n \geq 0} p(n)q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)^2_n}
\]
modular ✓ combinatorial ✓ $q$-hypergeometric ✓

\[
f(q) = \sum_{n \geq 0} (p_e(n) - p_0(n))q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)^2_n}
\]
modular ??? combinatorial ✓ $q$-hypergeometric ✓
Question

*What roles do mock theta functions play within the theory of modular forms?*
Modular forms and mock theta functions

Let $\tau \in \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$, and $q = e^{2\pi i \tau}$. 
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Mock and quantum modular forms

Modular forms and mock theta functions

Ramanujan’s last letter (1920).

- Asymptotics, near roots of unity, of “Eulerian” (i.e. \(q\)-hypergeometric) series.
Modular forms and mock theta functions

Ramanujan’s last letter (1920).

- Asymptotics, near roots of unity, of “Eulerian” (i.e. $q$-hypergeometric) series.

\[
\sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} = 1 + \frac{q}{(1 + q)^2} + \frac{q^4}{(1 + q)^2(1 + q^2)^2} + \cdots
\]
Question (Ramanujan)

Must Eulerian series with “similar asymptotics” be the sum of a modular form and a function which is \(O(1)\) (i.e. bounded) at all roots of unity?
Mock and quantum modular forms

Modular forms and mock theta functions

Question (Ramanujan)

Must Eulerian series with “similar asymptotics” be the sum of a modular form and a function which is $O(1)$ (i.e. bounded) at all roots of unity?

Ramanujan’s Answer

“The answer is it is not necessarily so.... I have not proved rigorously that it is not so. But I have constructed a number of examples...”
Define the modular form $b(q)$ by

$$b(q) := \prod_{m \geq 1} (1 - q^{2m-1}) \times \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}$$
Define the modular form \( b(q) \) by

\[
b(q) := \prod_{m \geq 1} (1 - q^{2m-1}) \times \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)} \quad (q = e^{2\pi i \tau})
\]
Mock and quantum modular forms

Modular forms and mock theta functions

Claim (Ramanujan)

If $q$ radially approaches an even order $2k$ root of unity, then

$$f(q) - (-1)^k b(q) = O(1).$$
Mock and quantum modular forms

F. Dyson (1987)

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future..."
F. Dyson (1987)

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...Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi.

...This remains a challenge for the future...”
Theorem (Zwegers, ‘02)

Ramanujan’s mock theta functions may be “completed” to obtain vector valued, non-holomorphic modular forms.
A completion

Example:

\[ F(\tau) := (q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}), 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}))^T \]
A completion

Example:

\[ F(\tau) := \left( q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}), 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}) \right)^T \]

\[ f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2} \]

\[ \omega(q) := \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(q; q^2)_n^2} \]
A completion

Example (cont.):

\[ G(\tau) := 2i\sqrt{3}\left( \int_{-\tau}^{i\infty} \frac{g_1(z)}{\sqrt{-i(\tau + z)}} \, dz, \int_{-\tau}^{i\infty} \frac{g_0(z)}{\sqrt{-i(\tau + z)}} \, dz, \int_{-\tau}^{i\infty} \frac{-g_2(z)}{\sqrt{-i(\tau + z)}} \, dz \right)^T \]
A completion

Example (cont.):

\[ G(\tau) := 2i\sqrt{3} \left( \int_{-\infty}^{i\infty} \frac{g_1(z) \, dz}{\sqrt{-i(\tau + z)}}, \int_{-\infty}^{i\infty} \frac{g_0(z) \, dz}{\sqrt{-i(\tau + z)}}, \int_{-\infty}^{i\infty} \frac{-g_2(z) \, dz}{\sqrt{-i(\tau + z)}} \right)^T \]

\[ g_j(z) = \text{weight } \frac{3}{2} \text{ modular } \Theta-\text{functions} \]
Some results of Zwegers

**Theorem (Zwegers)**

*The function*

\[ H(\tau) := F(\tau) - G(\tau) \]

*is a vector-valued weight \( \frac{1}{2} \) non-holomorphic modular form. In particular, \( H(\tau) \) transforms as*

\[
H(\tau + 1) = \begin{pmatrix}
\zeta_24^{-1} & 0 & 0 \\
0 & 0 & \zeta_3 \\
0 & \zeta_3 & 0 \\
\end{pmatrix}
H(\tau),
\]

\[
H(-1/\tau) = \sqrt{-i\tau}
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1 \\
\end{pmatrix}
H(\tau).
\]
Harmonic Maass forms

Definition (Bruinier-Funke, ’04)

A harmonic Maass form of weight $k \in \frac{1}{2} \mathbb{Z}$ on $\Gamma_0(4N)$ is a smooth $M: \mathcal{H} \to \mathbb{C}$ satisfying

1. Transformation law:
   $\forall A \in \Gamma_0(4N), \tau \in \mathcal{H}$, $M(A \tau) = \left\{ \begin{array}{ll}
   \frac{c \tau + d}{c \tau}^k M(\tau), & k \in \frac{1}{2} \mathbb{Z} - \mathbb{Z} \\
   \frac{d}{c \tau}^k M(\tau), & k \in \mathbb{Z}
   \end{array} \right.$

2. Harmonic:
   $\Delta_k M = 0$, where
   $\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$

3. $M$ satisfies a suitable growth condition in the cusps.
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1. **transformation law:** $\forall A \in \Gamma_0(4N), \tau \in \mathbb{H},$

   $$M(A\tau) = \begin{cases} 
   \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (c\tau + d)^k M(\tau), & k \in \frac{1}{2} \mathbb{Z} - \mathbb{Z} \\
   (c\tau + d)^k M(\tau), & k \in \mathbb{Z} 
   \end{cases}$$

2. **harmonic:** $\Delta^k M = 0$, where $\Delta^k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

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   $$
   M(A\tau) = \begin{cases} 
   (\frac{c}{d})^{2k} e^{-2k} (c\tau + d)^k M(\tau), & k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\
   (c\tau + d)^k M(\tau), & k \in \mathbb{Z}
   \end{cases}
   $$

2. **harmonic:** $\Delta_k M = 0$, where

   $$
   \Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
   $$

   ($\tau = x + iy$)
Harmonic Maass forms

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A **harmonic Maass form** of weight \( k \in \frac{1}{2} \mathbb{Z} \) on \( \Gamma_0(4N) \) is a smooth \( M : \mathbb{H} \to \mathbb{C} \) satisfying

1. **transformation law:** \( \forall A \in \Gamma_0(4N), \tau \in \mathbb{H} \),

\[
M(A\tau) = \begin{cases} 
  \left( \frac{c}{d} \right)^{2k} \varepsilon_d^{-2k} (c\tau + d)^k M(\tau), & k \in \frac{1}{2} \mathbb{Z} - \mathbb{Z} \\
  (c\tau + d)^k M(\tau), & k \in \mathbb{Z}
\end{cases}
\]

2. **harmonic:** \( \Delta_k M = 0 \), where

\[
\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

3. \( M \) satisfies a suitable **growth condition** in the cusps.
Technical remarks.

1. \( \Gamma_0(N) := \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \} \),

Kronecker symbol: \( \left( \frac{\cdot}{\cdot} \right) \)

\[ \epsilon_d := \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4} \\ i, & \text{if } d \equiv 3 \pmod{4} \end{cases} \]

We take the principal branch of the holomorphic \( \sqrt{\cdot} \).
Harmonic Maass forms

Technical remarks.

1. \[ \Gamma_0(N) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}, \]

2. We require \( 4 \mid N \) if \( k \in \frac{1}{2} \mathbb{Z} - \mathbb{Z} \).
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Harmonic Maass forms

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Harmonic Maass forms

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4. $\epsilon_d := \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}, \\ i, & \text{if } d \equiv 3 \pmod{4}. \end{cases}$

5. We take the principal branch of the holomorphic $\sqrt{\cdot}$. 
Condition (3): \( \exists \ P_M \in \mathbb{C}[q^{-1}] \) such that

\[ M(\tau) - P_M(\tau) = O(e^{-\epsilon y}), \]

for some \( \epsilon > 0 \) as \( y \to \infty \).
Mock and quantum modular forms

Harmonic Maass forms

Lemma

Harmonic Maass forms $M$ decompose into two parts:

\[ M = M_+ + M_- \]

where

$M_+ := \sum_{n \geq r} M_{c+}(n)q^n$ \hspace{1cm} \text{holomorphic part}$\quad (r_M \in \mathbb{Z})$

$M_- := \sum_{n < 0} c^- M(n) \Gamma(1-k, 4\pi |n| y)q^n$ \hspace{1cm} \text{non-holomorphic part}$\quad \Gamma(a, x) := \int_{\infty}^{x} t^{a-1} e^{-t} \, dt$
Harmonic Maass forms

**Lemma**

*Harmonic Maass forms* $M$ *decompose into two parts:*

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Harmonic Maass forms

Lemma

Harmonic Maass forms \( M \) decompose into two parts:

\[
M = M^+ + M^-,
\]

where

\[
M^+ := \sum_{n \geq r_M} c_M^+(n)q^n \quad \text{“holomorphic part”} \quad (r_M \in \mathbb{Z})
\]

\[
M^- := \sum_{n < 0} c_M^-(n)\Gamma(1-k, 4\pi|n|y)q^n \quad \text{“non-holomorphic part”}
\]
Harmonic Maass forms

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Harmonic Maass forms

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Fact: Ramanujan’s mock theta functions are examples of “holomorphic parts” $M^+$ of harmonic Maass forms.
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Ex. $M^+(\tau) = q^{-\frac{1}{24}} \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2}$
Fact: Ramanujan’s mock theta functions are examples of “holomorphic parts” $M^+$ of harmonic Maass forms.

**Ex.** $M^+(\tau) = q^{-\frac{1}{24}} \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2}$

**Definition (Zagier).** A mock modular form := a holomorphic part of a HMF.
Applications

- $q$-series
- modular $L$-functions
- combinatorics
- generalized Borcherds products
- Moonshine
- Donaldson invariants
- mathematical physics
- quantum modular forms
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- $q$-series
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  ...
Quantum modular forms

Let \( \tau \in \mathbb{H} := \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \), and \( q = e^{2\pi i \tau} \).
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\[ q = e^{2\pi i \tau} \]

- $\tau \in \mathbb{H}^+ \iff |q| < 1$
- $\tau \in \mathbb{H}^- \iff |q| > 1$
Quantum modular forms

Let $g : \mathbb{H} \to \mathbb{C}$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \text{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$. 

Quantum modular forms

Let \( g : \mathbb{H} \to \mathbb{C} \), \( \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \text{SL}_2(\mathbb{Z}), \quad \tau \in \mathbb{H} \).

**Modular transformation:**

\[
g \left( \frac{a\tau + b}{c\tau + d} \right) = \varepsilon(\gamma)(c\tau + d)^k g(\tau)
\]

or rather,

\[
g(\tau) - \varepsilon(\gamma)^{-1}(c\tau + d)^{-k} g \left( \frac{a\tau + b}{c\tau + d} \right) = 0
\]
Quantum modular forms

Let $g : \mathbb{Q} \rightarrow \mathbb{C}$, $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \text{SL}_2(\mathbb{Z})$, $x \in \mathbb{Q}$.

Modular transformation:

$$g(x) - \epsilon(\gamma)^{-1}(cx + d)^{-k}g \left( \frac{ax + b}{cx + d} \right) = ?$$
Definition (Zagier ’10)

A quantum modular form of weight $k$ ($k \in \frac{1}{2} \mathbb{Z}$) is function $g : \mathbb{Q} \setminus S \to \mathbb{C}$, for some discrete subset $S$, such that
Quantum modular forms

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Quantum modular forms

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\[
    h_{\gamma}(x) = h_{g,\gamma}(x) := g(x) - \varepsilon(\gamma)^{-1}(cx + d)^{-k}g \left( \frac{ax + b}{cx + d} \right)
\]

satisfy a suitable property of continuity or analyticity in \( \mathbb{R} \).
Example. Kontsevich’s “strange” function \((x \in \mathbb{Q})\):

\[
\phi(x) := e^{\pi ix \over 12} \sum_{n=0}^{\infty} (e^{2\pi ix}, e^{2\pi ix})_n
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Example. Kontsevich’s “strange” function \( \phi(x) = e^{\pi i x/12} \sum_{n=0}^{\infty} (e^{2 \pi i x}; e^{2 \pi i x})_n \).

Note. \( \phi \) converges for no open subset of \( \mathbb{C} \).
Quantum modular forms

**Example.** Kontsevich’s “strange” function ($x \in \mathbb{Q}$):

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\phi(x) := e^{\frac{\pi i x}{12}} \sum_{n=0}^{\infty} \left( e^{2\pi i x}, e^{2\pi i x} \right)_n
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**Note.** $\phi$ converges for no open subset of $\mathbb{C}$.

**Exercise:** $\phi$ converges for any rational number $x$. 
Quantum modular forms

Theorem (Zagier)

The function $\phi$ is a quantum modular form of weight $3/2$, i.e.

$$
\phi(x + 1) = \zeta_24 \phi(x),
$$

$$
\phi(x) \pm \zeta_8 \left| x \right|^{-3/2} \phi(-1/x) = h(x),
$$

where $h$ is a real analytic function (except at 0).
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**where** $h$ **is a real analytic function (except at 0).**
A sequence \( \{a_j\}_{j=1}^s \) of integers is called *strongly unimodal* of size \( n \) if

\[
a_1 + a_2 + \cdots + a_s = n,
\]

where

\[
a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots > a_s > 0
\]

for some \( r \).

The rank equals \( s - 2r + 1 \) (difference between # terms after and before the "peak").

Example:

\[
\{2, 5, 8, 1\}
\]

is a s.u.s. of size 2 + 5 + 8 + 1 = 16 and rank 4 - 2 · 3 + 1 = -1.

\[
\{1, 2, 3, 5, 3, 2\}
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is also a s.u.s. of size 16 and rank 6 - 2 · 4 + 1 = -1.
A sequence \( \{a_j\}_{j=1}^s \) of integers is called \textit{strongly unimodal} of size \( n \) if

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### Definition

A sequence $\{a_j\}_{j=1}^s$ of integers is called *strongly unimodal* of size $n$ if

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---

**Example**

- $\{2, 5, 8, 1\}$ is a s.u.s. of size $2 + 5 + 8 + 1 = 16$ and rank $4 - 2 \cdot 3 + 1 = -1$.
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Mock and quantum modular forms
Quantum modular forms

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Mock and quantum modular forms

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Quantum modular forms

A combinatorial generating function.

\[ u(m, n) := \# \{ \text{size } n \text{ strongly unimodal sequences of rank } m \}. \]
Quantum modular forms

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\[ U(\tau) := q^{-\frac{1}{24}} \sum_{n=0}^{\infty} (q; q)^2_n q^{n+1} = q^{-\frac{1}{24}} \sum_{\substack{m \in \mathbb{Z} \\ n \geq 1}} u(m, n)(-1)^m q^n. \]
Theorem (Bryson-Ono-Pittman-Rhoades)

Let $x \in \mathbb{Q}$. We have that

$$\phi(-x) = U(x).$$
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For $x \in \mathbb{H} \cup \mathbb{Q}$, the function $U$ satisfies

$$U(x) + (-ix)^{-\frac{3}{2}} U(-1/x) = h(x),$$

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Quantum modular forms

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Modular forms and mock theta functions

Ramanujan’s claim revisited (F - Ono - Rhoades)
Claim (Ramanujan)

If $q$ radially approaches an even order $2k$ root of unity, then

$$f(q) - (-1)^k b(q) = O(1).$$
Claim (Ramanujan)

If $q$ radially approaches an even order $2k$ root of unity, then

$$f(q) − (-1)^k b(q) = O(1).$$

$$b(q) := \prod_{m \geq 1} (1 - q^{2m-1}) \times \sum_{n \in \mathbb{Z}} (-1)^n q^n = q^{\frac{1}{24}} \frac{\eta^3(\tau)}{\eta^2(2\tau)}$$

$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q; q)_n^2}$$
Numerics

As $q \to -1$, we computed

\[ f(-0.994) \sim -1 \cdot 10^{31}, \quad f(-0.996) \sim -1 \cdot 10^{46}, \quad f(-0.998) \sim -6 \cdot 10^{90} \ldots \]
Ramanujan’s claim gives:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$-0.990$</th>
<th>$-0.992$</th>
<th>$-0.994$</th>
<th>$-0.996$</th>
<th>$-0.998$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(q) + b(q)$</td>
<td>3.961...</td>
<td>3.969...</td>
<td>3.976...</td>
<td>3.984...</td>
<td>3.992...</td>
</tr>
</tbody>
</table>
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\[
\begin{array}{|c|c|c|c|c|c|}
\hline
q & -0.990 & -0.992 & -0.994 & -0.996 & -0.998 \\
\hline
f(q) + b(q) & 3.961\ldots & 3.969\ldots & 3.976\ldots & 3.984\ldots & 3.992\ldots \\
\hline
\end{array}
\]

This suggests that

\[
\lim_{q \to -1} (f(q) + b(q)) = 4.
\]
Mock and quantum modular forms

### Numerics (cont.)

<table>
<thead>
<tr>
<th>$q$</th>
<th>0.992$i$</th>
<th>0.994$i$</th>
<th>0.996$i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(q)$</td>
<td>$2 \cdot 10^6 - 4.6 \cdot 10^6$i</td>
<td>$2 \cdot 10^8 - 4 \cdot 10^8$i</td>
<td>$1.0 \cdot 10^{12} - 2 \cdot 10^{12}$i</td>
</tr>
<tr>
<td>$f(q) - b(q)$</td>
<td>$\sim 0.05 + 3.85$i</td>
<td>$\sim 0.04 + 3.89$i</td>
<td>$\sim 0.03 + 3.92$i</td>
</tr>
</tbody>
</table>

This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4.64$$
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$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$
Questions

What are the $O(1)$ constants in

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = O(1)?$$

How do they arise?
Theorem (F-Ono-Rhoades)

If $\zeta$ is an even $2k$ order root of unity, then

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = -4 \sum_{n=0}^{k-1} (1 + \zeta)^2 (1 + \zeta^2)^2 \cdots (1 + \zeta^n)^2 \zeta^{n+1}.$$
**Remark.** We prove this as a special case of a more general theorem involving:

\[
R(w; q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} N(m, n) w^m q^n = \sum_{n=0}^\infty q^n (wq; q) \frac{N(w; q)}{(w-1q; q)^n},
\]

**mock modular rank function** [Bringmann-Ono]

\[
C(w; q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} M(m, n) w^m q^n = (q; q) \frac{C(wq; q)}{(w; q)(w-1q; q)},
\]

**modular crank function**

\[
U(w; q) := \sum_{n \geq 1} \sum_{m \in \mathbb{Z}} u(m, n) (-w)^m q^n = \sum_{n=0}^\infty (wq; q) \frac{U(w; q)}{(w-1q; q)^n},
\]

**quantum modular unimodal function** [B-O-P-R]
Radial limits

**Remark.** We prove this as a special case of a more general theorem involving:

\[ R(w; q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} N(m, n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n}, \]

mock modular rank function [Bringmann-Ono]

\[ C(w; q) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} M(m, n) w^m q^n = \frac{(q; q)_\infty}{(wq; q)_\infty (w^{-1}q; q)_\infty}, \]

modular crank function

\[ U(w; q) := \sum_{n \geq 1} \sum_{m \in \mathbb{Z}} u(m, n)(-w)^m q^n = \sum_{n=0}^{\infty} (wq; q)_n (w^{-1}q; q)_n q^{n+1}. \]

quantum modular unimodal function [B-O-P-R]
Combinatorial “modular” forms

Here,

\[ \mathcal{N}(m, n) := \# \{ \text{partitions } \lambda \text{ of } n \mid \text{rank}(\lambda) = m \}, \]

\[ \mathcal{M}(m, n) := \# \{ \text{partitions } \lambda \text{ of } n \mid \text{crank}(\lambda) = m \}, \]

\[ u(m, n) := \# \{ \text{size } n \text{ strongly unimodal sequences with rank } m \}. \]
Theorem (F-Ono-Rhoades)

If $\zeta_b = e^{\frac{2\pi i}{b}}$ and $1 \leq a < b$, then for every suitable root of unity $\zeta$ there is an explicit integer $c$ for which

$$
\lim_{q \to \zeta} \left( R(\zeta_b^a; q) - \zeta_b^c C(\zeta_b^a; q) \right) = -(1 - \zeta_b^a)(1 - \zeta_b^{-a}) U(\zeta_b^a; \zeta).
$$
Remark

The first theorem is the special case $a = 1$, $b = 2$, using the facts that

$$R(-1; q) = f(q) \quad \text{and} \quad C(-1; q) = b(q).$$
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\]

Remark

With the specialization \((w; q) \mapsto (\zeta^a_b; \zeta)\), the function \( U(\zeta^a_b; \zeta) \) is a finite sum.
Radial limits

Theorems \[ \iff \]

\[ \lim_{q \to \zeta} (\text{Mock } \vartheta - \text{Mod. Form}) = \text{Quantum MF}. \]
Radial limits

Theorems $\implies$

$$\lim_{q \to \zeta} (\text{Mock } \vartheta - \text{Mod. Form}) = \text{Quantum MF}.$$ 

$$\lim_{q \to \zeta} (\text{rank function} - \text{crank function}) = \text{unimodal function.}$$
Mock and quantum modular forms

Upper and lower half-planes

For Ramanujan's $f(q)$, remarkably we have

$$f(q - 1) = \sum_{n=0}^{\infty} q^n (-q; q)_2^n = 1 + q - q^2 + 2q^3 - 4q^4 + \ldots$$

Remark. Under $\tau \leftrightarrow q = e^{2\pi i \tau}$, $f(q)$ is defined on both $H \cup H^{-}$. 
Mock and quantum modular forms

Upper and lower half-planes

Proof philosophy.

Example

For Ramanujan’s $f(q)$, remarkably we have

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Remark.

Under $\tau \leftrightarrow q = e^{2\pi i \tau}$, $f(q)$ is defined on both $\mathbb{H} \cup \mathbb{H}^-$. 

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Mock and quantum modular forms

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Mock and quantum modular forms
Upper and lower half-planes

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Mock and quantum modular forms

Upper and lower half-planes

A larger framework:

Mock modular forms

Quantum modular forms

False theta functions
Your Hit Parade: The Top Ten Most Fascinating Formulas in Ramanujan’s Lost Notebook

George E. Andrews and Bruce C. Berndt

At 7:30 on a Saturday evening in March 1956, the first author sat down in an easy chair in the living room of his parents’ farm home ten miles east of Salem, Oregon, and turned the TV channel knob to NBC’s Your Hit Parade to find out the Top Seven Songs of the week, as determined by a national “survey” and sheet music sales. Little did this teenager know that almost exactly twenty years later, he would be at Trinity College, Cambridge, to discover one of the biggest “hits” in mathematical history, Ramanujan’s Lost Notebook. Meanwhile, at that same hour on that same Saturday night in Stevensville, Michigan, but at 9:30, the second author sat down in an overstuffed chair in front of the TV in his parents’ farm home anxiously waiting to learn the identities of the Top Seven Songs, sung by Your Hit Parade singers, Russell Arms, Dorothy Collins (his favorite singer), Snooky Lanson, and Gisele MacKenzie. About twenty years later, that author’s life would begin to be consumed by Ramanujan’s mathematics, but more important than Ramanujan to him this evening was how long his parents would allow him to stay up to watch Saturday night wrestling after Your Hit Parade ended. Just as the authors anxiously waited for the identities of the Top Seven Songs of the week years ago, readers of this article must now be brimming with unbridled excitement to learn the identities of the Top Ten Most Fascinating Formulas from Ramanujan’s Lost Notebook. The choices for the Top Ten Formulas were made by the authors. However, motivated by the practice of Your Hit Parade, but now extending the “survey” outside the boundaries of the U.S., we have taken an international “survey” to determine the proper order of fascination and amazement of these formulas. The survey panel of 34 renowned experts on Ramanujan’s work includes Nandadeep Deka Baruah, S. Bhargava, Jonathan Borwein, Peter Borwein, Douglas Bowman, David Bradley, Kathrin Bringmann, Song Heng Chan, Robin Chapman, Youn See Choi, Wenchang Chu, Shaun Cooper, Sylvie Corteel, Freeman Dyson, Ronald Evans, Philippe Flajolet, Christian Krattenthaler, Zhi-Guo Liu, Lisa Lorentzen, Jeremy Lovejoy, Jimmy McLaughlin, Steve Milne, Kent Ono, Peter Paule, Moxan Rahman, Anne Schilling, Michael Schlosser, Andrew Sills, Jaebum Sohn, S. Ole Warnaar, Kenneth Williams, Ae Ja Yee, Alexandru Zaharescu, and Doron Zeilberger. A summary of their rankings can be found in the last section of our paper. Just as the songs changed...
Poll to rank most fascinating formulas in the “lost” notebook:

1. Dyson’s Ranks.
2. Mock $\vartheta$-functions.
4. Continued fraction with three limit points.
5. Early QMFs: “Sums of Tails” of Euler’s Products.
Mock and quantum modular forms

Related work:

- Bajpai-Kimport-Liang-Ma-Ricci
- Bringmann-Creutzig-Rolen
- Bringmann-F-Rhoades
- Bringmann-Rolen
- Bryson-Ono-Pittman-Rhoades
- F-Ki-Truong Vu-Yang
- F-Ono-Rhoades
- Hikami-Lovejoy
- Joo-Löbrich
- Rolen-Schneider
- Zagier
- etc.
Thank you

\[ f(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2} \]

\[ f(q^{-1}) = \sum_{n \geq 0} \frac{q^n}{(-q;q)_n^2} \]