Hypergeometric Functions over Finite Fields

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Joint work with Jenny Fuselier, Ravi Ramakrishna, Holly Swisher and Fang-Ting Tu, arXiv:1510.02575

Overview

- Classical hypergeometric functions are well-understood. They are related to
  - ◇ periods of algebraic varieties
  - ◇ comb. identities and orthogonal poly.s
  - ◇ (arithmetic) triangle groups
  - ◇ $\cdots$
- Hypergeometric functions over finite fields are developed by Evans, Greene, Katz, McCarthy, Ono, $\cdots$
  - ◇ computing L-functions of algebraic varieties
  - ◇ proving supercongruences (Apéry or Ramanujan types)
  - ◇ obtaining character sum identities and estimate
  - ◇ computing arithmetic invariants of hypergeometric varieties

Hypergeometric Functions $\Leftrightarrow$ Finite Hypergeometric Functions

Hyper. Varieties or Hyper. Motives

Motivations and applications

- $GL(2)$-type Galois representations and automorphic forms (Li-Liu-L.)
- 2-dim'l abelian varieties admitting quaternionic multiplication (QM) (Deines-Fuselier-L.-Swisher-Tu)
- L-functions of algebraic varieties and related supercongruences (Deines-Fuselier-L.-Swisher-Tu)
- Characterization of intersecting families of maximum size in $PSL(2, q)$ (L.-Plaza-Sin-Xiang)
- Translating the symmetries of hypergeometric functions to finite hypergeometric functions (Fuselier-L.Ramakrishna-Swisher-Tu)

Notation in the classical setting

Gamma function

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t.$$

Satisfying

$\Gamma(x+1)/\Gamma(x) = x$ if $x \notin \mathbb{Z}_{\leq 0}$

$\Gamma(n+1) = n!$ when $n \in \mathbb{N}$.

Reflection formula.

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}$$

Multiplication formula, e.g. duplication formula

$$\Gamma(2a)\Gamma\left(\frac{1}{2}\right) = 2^{2a-1}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right), \forall a \in \mathbb{C}.$$

Notation in the classical setting

- Beta function

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

- $(a)_k := a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$
- Binomial theorem

$$(1-x)^{-a} = \sum_{k=0}^{\infty} \binom{-a}{k} (-x)^k = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} x^k.$$

- Use ζ_N to denote a primitive Nth root of unity.

Classical ${}_2F_1$ functions

Definition

For fixed parameters a, b, c and argument z, let

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ;\ z\right] := 1 + \sum_{k\geq 1} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}. $$

It satisfies an order-2 ordinary differential equation

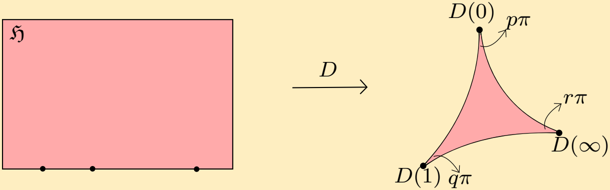
$$ HDE(a, b; c; z) : z(1-z)F'' + [(a+b+1)z - c]F' + abF = 0, $$

with 3 regular singularities at 0, 1, and ∞ with a, b playing symmetric roles.

There are generalized hypergeometric functions ${}_{n+1}F_n$ which can be defined similarly and recursively.

Theorem (Schwarz)

Let f, g be two independent solutions to the differential equation $HDE(a, b; c; z)$ at a point $z \in \mathfrak{H}$, and let $p = |1-c|$, $q = |c-a-b|$, and $r = |a-b|$. Then the Schwarz map $D = f/g$ gives a bijection from $\mathfrak{H} \cup \mathbb{R}$ onto a curvilinear triangle, denoted by $\Delta(p, q, r)$, with vertices $D(0)$, $D(1)$, $D(\infty)$ and corresponding angles $p\pi$, $q\pi$, $r\pi$, as illustrated below.

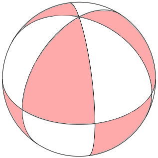


Uniformization Theorem

Assume $a, b, c \in \mathbb{Q}$. The universal cover of $\Delta(p, q, r)$ is either the Euclidean plane, the unit sphere, or the hyperbolic plane, depending on whether $p + q + r = 1, > 1, < 1$ respectively.

Examples:

1. $(a, b, c) = (\frac{1}{2}, \frac{1}{2}, 1)$, $(p, q, r) = (0, 0, 0)$, a hyperbolic triangle, biholomorphic to the fundamental domain of $\Gamma(2)$
2. $(a, b, c) = (\frac{1}{6}, \frac{2}{3}; \frac{4}{3})$, $(p, q, r) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, a spherical triangle,



Some properties of the ${}_2F_1$ functions

Transformation formulas, e.g. Pfaff transform

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\, ;\ z\right] = (1-z)^{-a}\, {}_2F_1\left[\begin{matrix} a & c-b \\ & c \end{matrix}\, ;\ \frac{z}{z-1}\right]. $$

Evaluation formulas, e.g. Gauss summation formula

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\, ;\ 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

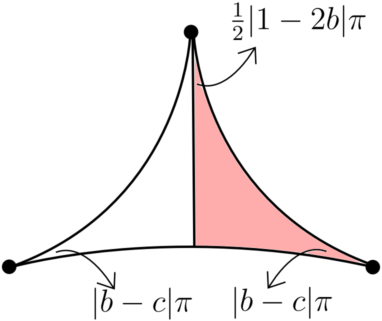
Algebraic identities, e.g.

$$ {}_2F_1\left[\begin{matrix} a & a-\frac{1}{2} \\ & 2a \end{matrix}\, ;\ z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a} $$

A quadratic transformation formula

$$ {}_2F_1\left[\begin{matrix} c & b \\ & c-b+1 \end{matrix} ; \; z\right] = (1-z)^{-c} \, {}_2F_1\left[\begin{matrix} \frac{c}{2} & \frac{1+c}{2}-b \\ & c-b+1 \end{matrix} ; \; \frac{-4z}{(1-z)^2}\right]. $$

The corresponding Schwarz triangle to the left, $\Delta(|b-c|, |b-c|, |1-2b|)$, can be tiled by two copies of the Schwarz triangle $\Delta(|b-c|, \frac{1}{2}, \frac{1}{2}|1-2b|)$ to the right.



The ${}_2P_1$ period functions

For $a, b, c \in \mathbb{C}$ with $\mathrm{Re}(c) > \mathrm{Re}(b)$ (can be relaxed), define

$$ {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] := \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\mathrm{d}x $$

Note that a, b play asymmetric roles.

Euler integral formula

$$
\begin{aligned}
{}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}\,; z\right] &:= \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\mathrm{d}x \\
&\overset{\text{binomial}}{=} \int_0^1 x^{b-1}(1-x)^{c-b-1}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}(zx)^k\mathrm{d}x \\
&= \sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!}\int_0^1 x^{b-1+k}(1-x)^{c-b-1}\mathrm{d}x \\
&\overset{\text{Beta}}{=} \sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!}B(b+k,c-b) \\
&= B(b,c-b)\,{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\,;\, z\right].
\end{aligned}
$$

Two essential ingredients

- The Beta function

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

- Binomial Theorem

$$(1-x)^{-a}=\sum_{k=0}^{\infty}\binom{-a}{k}(-1)^k x^k=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}x^k.$$

Hypergeometric functions as normalized period functions

Consider

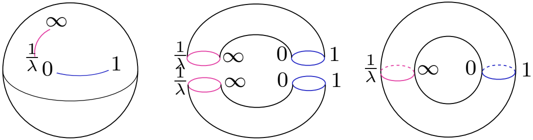
$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ;\ z\right] = \frac{1}{B(b, c-b)}\, {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; z\right] $$

as the normalized period function.

Note that $B(b, c-b)$ is also the value of ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ;\ 0\right]$.

The Legendre curves $E_\lambda : y^2 = x(1-x)(1-\lambda x)$

- E_λ is a double over of $\mathbb{C}P^1$ which ramifies at $0, 1, \frac{1}{\lambda}, \infty$.



- Holomorphic differential: $\omega_\lambda := \frac{\mathrm{d}x}{y} = \frac{\mathrm{d}x}{(x(1-x)(1-\lambda x))^{1/2}}$
- Periods: $\tau_1 := 2\int_0^1 \omega_\lambda = 2\pi \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda\right], \tau_2 = \int_{\gamma_2} \omega_\lambda$
- CM criterions: i) values of $j(E_\lambda)$; ii) $\tau_1/\tau_2 \in \overline{\mathbb{Q}}, \cdots$

Galois representations arising from E_λ, $\lambda \in \mathbb{Q} \setminus \{0,1\}$.

- Galois representations. For any fixed prime ℓ, from the torsion points $E_\lambda[\ell^n] \cong (\mathbb{Z}/\ell^n\mathbb{Z})^2$ permuting by $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, taking inverse limit, and enlarging coefficients, one obtains a continuous homomorphism

$$\rho_{E_\lambda,\ell} : G_\mathbb{Q} \to GL_2(\mathbb{Q}_\ell),$$

such that

$$\mathrm{Tr}\rho_{E_\lambda,\ell}(\mathrm{Frob}_p) = p + 1 - \#(E_\lambda/\mathbb{F}_p),$$

where Frob_p is the geometric Frobenius at p.

- Modularity theorem, $\exists$ a wt-2 modular form $f_\lambda = \sum_{n\geq 1} a_n(f_\lambda)q^n$ s. t. for each prime $p \nmid N_\lambda$,

$$a_p(f_\lambda) = \mathrm{Tr}\rho_{E_\lambda,\ell}(\mathrm{Frob}_p).$$

Notation for finite fields

- $\mathbb{F}_q$: finite field, $q = p^e$ odd
- $\mathbb{F}_q^\times$: a cyclic group of size $q-1$.
- Let $\mathbb{F}_{q^s}$ denote the degree s extension field of $\mathbb{F}_q$.
- A multiplicative character χ is a homo. $\mathbb{F}_q^\times \to \mathbb{C}^\times$.
- $\widehat{\mathbb{F}_q^\times}$: cyclic group of all multiplicative characters on $\mathbb{F}_q^\times$
- ε: trivial character
- ϕ: quadratic character. For $x \in \mathbb{F}_q^\times$, $\phi(x) = 1$ iff $x = a^2$ for some $a \in \mathbb{F}_q^\times$, i.e. x is a quadratic residue. In particular, when $q = p$, $\phi(x) = \left(\frac{x}{p}\right)$, the Legendre symbol.
- $\chi(0) = 0$ for each $\chi \in \mathbb{F}_q^\times$

Computing $\#(E_\lambda/\mathbb{F}_q)$

Recall

$$E_\lambda : \quad y^2 = x(1-x)(1-\lambda x) := f_\lambda(x).$$

For $x \in \mathbb{F}_q$, the equation has $1 + \phi(f_\lambda(x)) = \begin{cases} 1 \\ 2 \\ 0 \end{cases}$ solutions for y. Thus the total number of $\mathbb{F}_q$-solutions equals

$$1 + \sum_{x \in \mathbb{F}_q} (1 + \phi(f_\lambda(x))) = 1 + q + \sum_{x \in \mathbb{F}_q} \phi(x(1-x)(1-\lambda x)).$$

Thus $\mathrm{Tr}\rho_{E_\lambda,\ell}(\mathrm{Frob}_p) = -\sum_{x \in \mathbb{F}_p} \phi(x(1-x)(1-\lambda x))$.

$$\mathrm{Tr}\rho_{E_\lambda,\ell}(\mathrm{Frob}_p) = -\sum_{x\in\mathbb{F}_p}\left(\frac{x(1-x)(1-\lambda x)}{p}\right).$$

In comparison,

$${}_2P_1\left[\begin{matrix} a & b \\ & c\end{matrix}\,;z\right] = \int_0^1 (x(1-x)(1-\lambda x))^{-1/2}dx.$$

It suggests the following correspondences

$$\begin{matrix} a=\frac{i}{N} & \leftrightarrow & A\in\widehat{\mathbb{F}_p^\times} \text{ of order } N \\ x^a & \leftrightarrow & A(x) \\ -a & \leftrightarrow & \overline{A} \\ \int_0^1 \quad \mathrm{d}x & \leftrightarrow & \sum_{x\in\mathbb{F}} \end{matrix}.$$

In this vein, we view characters like $\sum_{x\in\mathbb{F}_p}\left(\frac{x(1-x)(1-\lambda x)}{p}\right)$ as period functions over finite fields.

Two ingredients

Given multiplicative characters A,B corresponding to a, b

- Jacobi sum (Beta function):

$$J(A,B) = \sum_{x \in \mathbb{F}_q} A(x)B(1-x)$$

$$\left(B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \right)$$

- Binomial Theorem:

$$A(1-x) = \delta(x) + \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} J(A,\overline{\chi})\chi(x)$$

$$\left((1-x)^a = \sum_{k=0}^{\infty} \binom{a}{k} (-1)^k x^k. \right)$$

Finite period/hypergeometric functions

Given $A, B, C \in \widehat{\mathbb{F}_q^\times}$, define

- the finite period function: (slightly modified from Greene's version)

$$ {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix} ; \lambda; q\right] := \sum_{x \in \mathbb{F}_q} B(x) C\overline{B}(1-x)\overline{A}(1-\lambda x). $$

$$ \left({}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; z\right] := \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} \mathrm{d}x \right) $$

- the finite hypergeometric function: (slightly modified from McCarthy's version)

$$ {}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix} ; \lambda; q\right] := \frac{1}{J(B, C\overline{B})} {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix} ; \lambda; q\right] $$

$$ \left({}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; z\right] := \frac{1}{B(b, c-b)} {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix} ; z\right] \right) $$

Gauss Sums

Let Ψ be a fixed nontrivial additive character of $\mathbb{F}_q$. Define the Gauss sum of $A \in \widehat{\mathbb{F}_q^\times}$ as

$$g(A) := \sum_{x \in \mathbb{F}_q} A(x)\Psi(x).$$

Similar to $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, one has when AB is nontrivial,

$$J(A, B) = \frac{g(A)g(B)}{g(AB)}.$$

The Gauss sums satisfies reflection formula,

$$g(A)g(\overline{A}) = qA(-1), \forall A \in \widehat{\mathbb{F}_q^\times}, A \neq \varepsilon. \tag{1}$$

Duplication and other multiplication formula:

$$g(A)g(\phi A) = g(A^2)g(\phi)\overline{A}(4),$$

where ϕ is the quadratic character.

Dictionary between $\mathbb{C}$ and $\mathbb{F}_q$ settings

Remark

Gauss sums are finite field analogues of the Gamma function. Jacobi sums are finite field analogues of the beta function.

$$
\begin{array}{ccc}
a = \frac{i}{N} & \leftrightarrow & A \in \widehat{\mathbb{F}_q^\times} \text{ of order } N \\
x^a & \leftrightarrow & A(x) \\
x^{a+b} & \leftrightarrow & A(x)B(x) = AB(x) \\
-a & \leftrightarrow & \overline{A} \\
\Gamma(a) & \leftrightarrow & g(A) \\
B(a,b) & \leftrightarrow & J(A,B) \\
\int_0^1 \quad dx & \leftrightarrow & \sum_{x \in \mathbb{F}}
\end{array}
$$

Under the definitions and the dictionary, one can immediately obtain the finite field analogues of classical formulas, such as the Pfaff transformation.

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = (1-z)^{-a}\, {}_2F_1\left[\begin{matrix} a & c-b \\ & c \end{matrix}; \frac{z}{z-1}\right] $$

which can be obtained from the Euler integral formula and the binomial theorem.

The proof generalizes straightforwardly to the finite field verion:
For A, B, $C \in \widehat{\mathbb{F}_q^\times}$, and $\lambda \in \mathbb{F}_q, \lambda \neq 1$, we have

$$ {}_2\mathbb{F}_1 \begin{bmatrix} A & B \\ & C \end{bmatrix} ; \lambda \Bigg] = \overline{A}(1-\lambda)\ {}_2\mathbb{F}_1 \begin{bmatrix} A & \overline{B}C \\ & C \end{bmatrix} ; \frac{\lambda}{\lambda-1} \Bigg]. $$

$$ \left({}_2F_1 \left[\begin{matrix} a & b \\ & c \end{matrix} ; z \right] = (1-z)^{-a}\ {}_2F_1 \left[\begin{matrix} a & c-b \\ & c \end{matrix} ; \frac{z}{z-1} \right] \right) $$

The proof of

$$(1-z)^{-c}\,{}_2F_1\left[\begin{matrix}\frac{c}{2} & \frac{1+c}{2}-b \\ & c-b+1\end{matrix};\ \frac{-4z}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix}c & b \\ & c-b+1\end{matrix};\ z\right]$$

is mainly based on the binomial theorem and the properties of $\Gamma(x)$ including the duplication formula.

Theorem

Let $B, D \in \widehat{\mathbb{F}_q^\times}$, and set $C = D^2$. When $D \neq \phi$ and $B \neq D$, we have

$$\overline{C}(1-x)\ {}_2\mathbb{F}_1\left[\begin{matrix} D\phi\overline{B} & D \\ & C\overline{B} \end{matrix}\ ;\ \frac{-4x}{(1-x)^2}\right]$$

$$= {}_2\mathbb{F}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}\ ;\ x\right] - \delta(1-x)\frac{J(C,\overline{B}^2)}{J(C,\overline{B})} - \delta(1+x)\frac{J(\overline{B},D\phi)}{J(C,\overline{B})}.$$

The main difficult lies in analyzing the degenerate cases.

$$\left((1-z)^{-c}\ {}_2F_1\left[\begin{matrix} \frac{c}{2} & \frac{1+c}{2}-b \\ & c-b+1 \end{matrix}\ ;\ \frac{-4z}{(1-z)^2}\right] = {}_2F_1\left[\begin{matrix} c & b \\ & c-b+1 \end{matrix}\ ;\ z\right].\right)$$

This approach is particularly handy for translating to the finite field setting a classical transformation formula that satisfies the following condition:

> *($*$) it can be proved using only the binomial theorem, the reflection and multiplication formulas, or their corollaries.*

But not every formula satisfies this condition. The underlining geometry allows us to go beyond the ($*$) condition.

Consistent way to assign characters as q varies

Fix $N \geq 2$ and let $K = \mathbb{Q}(\zeta_N)$, and $\mathcal{O}_K$ be its ring of integers. Let $\mathfrak{p}$ be a finite prime ideal of $\mathcal{O}_K$, coprime to N and let $\mathbb{F}_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$ where $q_\mathfrak{p} := |\mathcal{O}_K/\mathfrak{p}| \equiv 1 \pmod{N}$.

Definition

For any given $i \in \mathbb{Z}$, $\iota_\mathfrak{p}\left(\frac{i}{N}\right) : \mathfrak{p} \to \widehat{\mathbb{F}_\mathfrak{p}^\times}$ as

$$\iota_\mathfrak{p}\left(\frac{i}{N}\right)(x) \equiv x^{i\cdot(|\mathcal{O}_K/\mathfrak{p}|-1)/N} \mod \mathfrak{p}.$$

Theorem

Let $a, b, c \in \mathbb{Q}$ with least common denominator N such that $a,\ b,\ a-c,\ b-c \notin \mathbb{Z}$ and $\lambda \in \mathbb{Q} \setminus \{0,1\}$. Let $K = \mathbb{Q}(\zeta_N)$ with the ring of integers $\mathcal{O}_K$, and let ℓ be any prime. Then there is a 2-dimensional representation $\sigma_{\lambda,\ell}$ of $G_K := Gal(\overline{K}/K)$ over $\mathbb{Q}_\ell(\zeta_N)$, depending on a, b, c, such that for each unramified prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ for which λ and $1-\lambda$ can be mapped to nonzero elements in the residue field, $\sigma_{\lambda,\ell}$ evaluated at the arithmetic Frobenius conjugacy class $Frob_\mathfrak{p}$ at $\mathfrak{p}$ is an algebraic integer (independent of the choice of ℓ), satisfying

$$Tr\,\sigma_{\lambda,\ell}(Frob_\mathfrak{p}) = -{}_2\mathbb{P}_1\begin{bmatrix}\iota_\mathfrak{p}(a) & \iota_\mathfrak{p}(b) \\ & \iota_\mathfrak{p}(c)\end{bmatrix}; \lambda; q(\mathfrak{p})\Bigg] . \qquad (2)$$

Key Ingredient

Generalized Legendre Curves (Wolfart and Archinard)

By definition, ${}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix};\ \lambda\right] = \int_0^1 \omega_\lambda$ with ω_λ being a period of

$$C_\lambda^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^k \quad \text{where}$$

$$N = \mathrm{lcd}(a,b,c), i = N(1-b), j = N(1+b-c), k = Na.$$

For example,

- $(a,b,c) = (\frac{1}{6}, \frac{1}{3}, \frac{5}{6})$, $(p,q,r) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{6})$, it corresponds to the triangle group $(3,6,6)$, which is arithmetic. For this case, $N = 6, i = 4, j = 3, k = 1$.
- $(a,b,c) = (\frac{1}{12}, \frac{1}{4}, \frac{5}{6})$, $(p,q,r) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{2})$, it corresponds to the arithmetic triangle group $(2,6,6)$. For this case, $N = 12, i = 9, j = 5, k = 1$.

- Assume: $1 \leq i, j, k < N, N \nmid i+j+k$, $\lambda \in \mathbb{Q} \setminus \{0, 1\}$
- $X_\lambda^{[N;i,k,j]}$: the smooth model;
- $J_\lambda^{[N;i,j,k]}$: its Jacobian variety .
- $End(J_\lambda^{[N;i,j,k]})$ contains $\mathbb{Z}[\zeta_N]$ due to $\zeta : (x, y) \mapsto (x, \zeta_N^{-1} y)$
- Differentials has a basis consisting of the form $\frac{x^{b_0}(1-x)^{b_1}(1-\lambda x)^{b_2}}{y^n} \mathrm{d}x$ with $0 \leq n \leq N-1, b_i \in \mathbb{Z}$.
- The period matrix can be written explicitly in terms of hypergeometric series.

- If $d \mid N$, $J_\lambda^{[d;i,j,k]}$ is isogeous to a subvariety of $J_\lambda^{[N;i,j,k]}$
- J_λ^{new}: the primitive part of $J_\lambda^{[N;i,j,k]}$ defined over $\mathbb{Q}$.
- $\dim J_\lambda^{new} = \varphi(N)$, the Euler number of N.
- Let $\rho_{\lambda,\ell}^{new} : G_{\mathbb{Q}} \to GL_{2\varphi(N)}(\overline{\mathbb{Q}}_\ell)$ be the ℓ-adic Galois representation arising from J_λ^{new}.
- For each unramified prime ideal $\mathfrak{p}$, compute $\rho_{\lambda,\ell}^{new}|_{G_{\mathbb{Q}(\zeta_N)}}(\mathrm{Frob}_{\mathfrak{p}})$ using character sums which can be written in terms of explicit finite period functions.
- Due to ζ, $\rho_{\lambda,\ell}^{new}|_{G_{\mathbb{Q}(\zeta_N)}}$ is a direct sum of 2-dimensional Galois representations, which also gives the matching in the final claim.

The Galois perspective gives important guidelines

To look for finite field analogue of

$$ {}_2F_1\left[\begin{matrix} a & a-\frac{1}{2} \\ & 2a \end{matrix} ; \ z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}, $$

it is tempting to use the dictionary directly. However, the Galois perspective tells us that won't work.

Theorem

Let $z \in \mathbb{F}_q^\times$, and $A \in \widehat{\mathbb{F}_q^\times}$ have order larger than 2. Then

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & A\phi \\ & A^2 \end{matrix} ; z\right] = \left(\frac{1+\phi(1-z)}{2}\right) \cdot \left(\overline{A}^2\left(\frac{1+\sqrt{1-z}}{2}\right) + \overline{A}^2\left(\frac{1-\sqrt{1-z}}{2}\right)\right).
$$

$$
\left({}_2F_1\left[\begin{matrix} a & a-\frac{1}{2} \\ & 2a \end{matrix} ; z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}\right).
$$

2-dimensional abelian varieties with QM

One of our motivations was to use J_λ^{new} to construct 2-dimensional abelian varieties whose endomorphism algebra contains a quaternion algebra.

For $(3,6,6)$: $\lambda \in \overline{\mathbb{Q}} \setminus \{0,1\}$, the primitive part of $X_\lambda^{[6;4,3,1]}$ gives a family of 2-dimensional abelian varieties parameterized by the Shimura curve for (3,6,6) [Deines, Fuselier, L. Swisher, Tu]. A different construction was given by Shiga-Petkova using Picard curves.

For $(2,6,6)$: $\lambda \in \overline{\mathbb{Q}} \setminus \{0,1\}$, the primitive part of $X_\lambda^{[12;9,5,1]}$ is a 4-dimensional abelian variety. It is natural to ask how it decomposes.

As an application of the finite quadratic formula, one has

Theorem

Let $\lambda \in \overline{\mathbb{Q}}$ such that $\lambda \neq 0, \pm 1$. Let $J_{\lambda,1}^{new}$ (resp. $J_{\lambda,2}^{new}$) be the primitive part of the Jacobian variety of $X_\lambda^{[6;4,3,1]}$ (resp. $X_\lambda^{[12;9,5,1]}$). Then

$$J^{new}_{\frac{-4\lambda}{(1-\lambda)^2},2} \sim J^{new}_{\lambda,1} \oplus J^{new}_{\lambda,1}$$

over some number field depending on λ.

What we discuss here can be generalized to other ${}_{n+1}F_n$ hypergeometric functions via recursion. Consequently, we obtains explicit algebraic models for varieties whose periods are the desired hypergeometric functions.

For hypergeometric motives over $\mathbb{Q}$, a different realization using toric varieties has been given by Beukers, Cohen, and Mellit.

Thank you!