The Voronoi formula and double Dirichlet series

Eren Mehmet Kıral[1] Fan Zhou[2]

[1]Texas A&M University

[2]Ohio State University

Connecticut Summer School In Number Theory, 12-14 August 2016

What is Voronoi summation formula?

The Voronoi summation formula is an identity of the form $A = B$

What is Voronoi summation formula?

The Voronoi summation formula is an identity of the form $A = B$ where

- A is a weighted sum of Fourier coefficients of an automorphic form, twisted by exponential sums (or simply additive characters)

What is Voronoi summation formula?

The Voronoi summation formula is an identity of the form $A = B$ where

- A is a weighted sum of Fourier coefficients of an automorphic form, twisted by exponential sums (or simply additive characters)
- B is another (dual) sum of Fourier coefficients weighted differently and twisted differently.

What is Voronoi summation formula?

The Voronoi summation formula is an identity of the form $A = B$ where

- A is a weighted sum of Fourier coefficients of an automorphic form, twisted by exponential sums (or simply additive characters)
- B is another (dual) sum of Fourier coefficients weighted differently and twisted differently.

The prototype: Let $f(z) = \sum_n a_f(n) n^{\frac{k-1}{2}} e(nz)$ be a weight k cusp form, and $(a, c) = 1$ integers with $c > 0$:

What is Voronoi summation formula?

The Voronoi summation formula is an identity of the form $A = B$ where

- A is a weighted sum of Fourier coefficients of an automorphic form, twisted by exponential sums (or simply additive characters)
- B is another (dual) sum of Fourier coefficients weighted differently and twisted differently.

The prototype: Let $f(z) = \sum_n a_f(n) n^{\frac{k-1}{2}} e(nz)$ be a weight k cusp form, and $(a, c) = 1$ integers with $c > 0$:

$$\sum_{n=1}^{\infty} a_f(n) e\left(\frac{an}{c}\right) w\left(\frac{n}{X}\right) = \frac{1}{c} \sum_{m=1}^{\infty} a_f(m) e\left(-\frac{\overline{a}m}{c}\right) W\left(\frac{m}{c^2/X}\right)$$

where w is a Schwartz function, $a\overline{a} \equiv 1 \pmod{c}$, and W is an integral transform of w.

Classical Voronoi Summation Formula

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$.

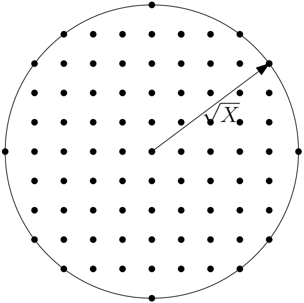
$$\sum_{n=0}^{\infty} r_2(n) w(n) = \sum_{m=0}^{\infty} r_2(m) \int_0^{\infty} w(x) \pi J_0(2\pi\sqrt{xm}) \mathrm{d}x.$$

Classical Voronoi Summation Formula

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$.

$$\sum_{n=0}^{\infty} r_2(n) w(n) = \sum_{m=0}^{\infty} r_2(m) \int_0^{\infty} w(x) \pi J_0(2\pi\sqrt{xm}) \mathrm{d}x.$$

Count the number of integral points inside a circle of growing radius

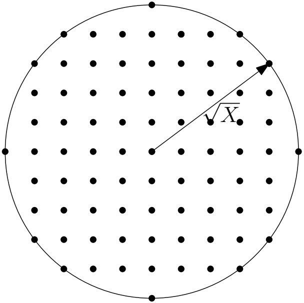


Classical Voronoi Summation Formula

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$.

$$\sum_{n=0}^{\infty} r_2(n) w(n) = \sum_{m=0}^{\infty} r_2(m) \int_0^{\infty} w(x) \pi J_0(2\pi\sqrt{xm}) \mathrm{d}x.$$

Count the number of integral points inside a circle of growing radius



Using the formula,

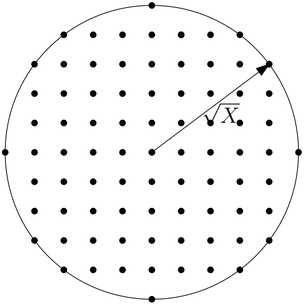
$$\sum_{\substack{a,b \in \mathbb{Z} \\ a^2+b^2 \le X}} 1 = \sum_{n \le X} r_2(n)$$

Classical Voronoi Summation Formula

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$.

$$\sum_{n=0}^{\infty} r_2(n) w(n) = \sum_{m=0}^{\infty} r_2(m) \int_0^{\infty} w(x) \pi J_0(2\pi\sqrt{xm}) \mathrm{d}x.$$

Count the number of integral points inside a circle of growing radius



Using the formula,

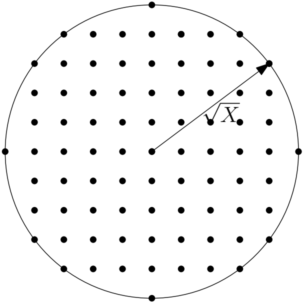
$$\sum_{\substack{a,b\in\mathbb{Z} \\ a^2+b^2\le X}} 1 = \sum_{n\le X} r_2(n) = \pi X + O(X^{\frac{1}{3}}).$$

Classical Voronoi Summation Formula

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$.

$$\sum_{n=0}^{\infty} r_2(n)w(n) = \sum_{m=0}^{\infty} r_2(m) \int_0^{\infty} w(x)\pi J_0(2\pi\sqrt{xm})\mathrm{d}x.$$

Count the number of integral points inside a circle of growing radius



Using the formula,

$$\sum_{\substack{a,b\in\mathbb{Z} \\ a^2+b^2\leq X}} 1 = \sum_{n\leq X} r_2(n) = \pi X + O(X^{\frac{1}{3}}).$$

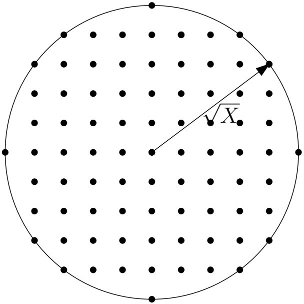
Decreasing the exponent $\frac{1}{3}$ is called the Gauss circle problem.

Classical Voronoi Summation Formula

Let $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$.

$$\sum_{n=0}^{\infty} r_2(n) w(n) = \sum_{m=0}^{\infty} r_2(m) \int_0^{\infty} w(x) \pi J_0(2\pi\sqrt{xm}) \mathrm{d}x.$$

Count the number of integral points inside a circle of growing radius



Using the formula,

$$\sum_{\substack{a,b\in\mathbb{Z} \\ a^2+b^2\le X}} 1 = \sum_{n\le X} r_2(n) = \pi X + O(X^{\frac{1}{3}}).$$

Decreasing the exponent $\frac{1}{3}$ is called the Gauss circle problem.

- By Hardy-Landau the exponent is $> \frac{1}{4}$.
- Huxley (2000) has lowered the exponent down to $\frac{131}{416} \approx 0.3149...$

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Is a function on $\mathrm{GL}(N, \mathbb{R})$, invariant on the left under $\mathrm{SL}(N, \mathbb{Z})$.

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$
- The Fourier coefficients satisfy the Hecke relations

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$
- The Fourier coefficients satisfy the Hecke relations

The L-function is given by

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$
- The Fourier coefficients satisfy the Hecke relations

The L-function is given by

$$L(s, F \qquad) = \sum_{n=1}^{\infty} \frac{A(1, \ldots, 1, n)}{n^s}$$

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$
- The Fourier coefficients satisfy the Hecke relations

The L-function is given by

$$L(s, F \times \chi^*) = \sum_{n=1}^{\infty} \frac{A(1, \ldots, 1, n)\chi^*(n)}{n^s}$$

GL_N Maass forms

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$
- The Fourier coefficients satisfy the Hecke relations

The L-function is given by

$$L(s, F \times \chi^*) = \sum_{n=1}^{\infty} \frac{A(1, \ldots, 1, n)\chi^*(n)}{n^s}$$

which satisfies a functional equation

$$L(s, F \times \chi^*) = \tau(\chi^*)^N c^{*-Ns} G(s) L(1-s, \widetilde{F} \times \overline{\chi^*})$$

We will generalize this to $\mathrm{GL}(N)$. A Hecke-Maass form F on $\mathrm{GL}(N)$:

- Has Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ indexed by integers $m_1, m_2, \ldots, m_{N-1}$
- The Fourier coefficients satisfy the Hecke relations

The L-function is given by

$$L(s, F \times \chi^*) = \sum_{n=1}^{\infty} \frac{A(1, \ldots, 1, n)\chi^*(n)}{n^s}$$

which satisfies a functional equation

$$L(s, F \times \chi^*) = \tau(\chi^*)^N c^{*-Ns} G(s) L(1-s, \widetilde{F} \times \overline{\chi^*})$$

where χ^* is a primitive Dirichlet character modulo c^* and G is some ratio of Gamma functions.

Statement of the formula

$$L_{\mathbf{q}}(s,F,a/c)=\sum_{n=1}^{\infty}\frac{A(q_{N-2},\ldots,q_1,n)e\left(\frac{\overline{a}n}{c}\right)}{n^s}$$

has an analytic continuation to all $s\in\mathbb{C}$ and satisfies the functional equation

$$L_{\mathbf{q}}(s,F,a/c)=\sum_{\pm}\frac{G_+(s)\mp G_-(s)}{2}\sum_{d_1|q_1c}\sum_{d_2|\frac{q_1q_2c}{d_1}}\cdots\sum_{d_{N-2}|\frac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-3}}}$$
$$\times\sum_{n=1}^{\infty}\frac{A(n,d_{N-2},\ldots,d_2,d_1)\,\mathrm{Kl}(a,\pm n,c;\mathbf{q},\mathbf{d})}{n^{1-s}c^{Ns-1}d_1d_2\cdots d_{N-2}}\frac{d_1^{(N-1)s}d_2^{(N-2)s}\cdots d_{N-2}^{2s}}{q_1^{(N-2)s}q_2^{(N-3)s}\cdots q_{N-2}^{s}}$$

Statement of the formula (continued)

and where the hyper-Kloosterman sum is,

$$\mathrm{Kl}(a,n,c;\mathbf{q},\mathbf{d}) = \sideset{}{^*}\sum_{x_1 (\mathrm{mod}\ \frac{q_1 c}{d_1})} \sideset{}{^*}\sum_{x_2 (\mathrm{mod}\ \frac{q_1 q_2 c}{d_1 d_2})} \cdots \sideset{}{^*}\sum_{x_{N-2} (\mathrm{mod}\ \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}})}$$

$$\times\, e\left(\frac{d_1 x_1 a}{c} + \frac{d_2 x_2 \overline{x_1}}{\frac{q_1 c}{d_1}} + \cdots + \frac{d_{N-2} x_{N-2} \overline{x_{N-3}}}{\frac{q_1 \cdots q_{N-3} c}{d_1 \cdots d_{N-3}}} + \frac{n \overline{x_{N-2}}}{\frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}}} \right),$$

Statement of the formula (continued)

and where the hyper-Kloosterman sum is,

$$\mathrm{Kl}(a, n, c; \mathbf{q}, \mathbf{d}) = \sideset{}{^*}\sum_{x_1 (\text{mod } \frac{q_1 c}{d_1})} \ \sideset{}{^*}\sum_{x_2 (\text{mod } \frac{q_1 q_2 c}{d_1 d_2})} \cdots \sideset{}{^*}\sum_{x_{N-2} (\text{mod } \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}})}$$

$$\times \, e\left(\frac{d_1 x_1 a}{c} + \frac{d_2 x_2 \overline{x_1}}{\frac{q_1 c}{d_1}} + \cdots + \frac{d_{N-2} x_{N-2} \overline{x_{N-3}}}{\frac{q_1 \cdots q_{N-3} c}{d_1 \cdots d_{N-3}}} + \frac{n \overline{x_{N-2}}}{\frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}}} \right),$$

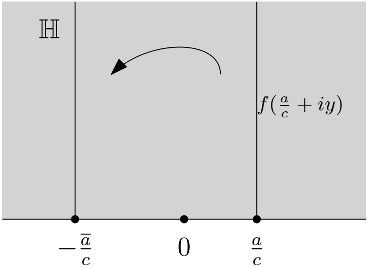
and where the function $G_{\pm}$ is a ratio of Gamma factors

The Golfeld-Li proof

Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.

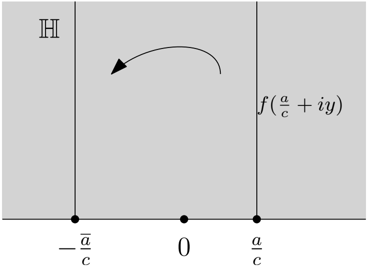
The Golfeld-Li proof

Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.



The Golfeld-Li proof

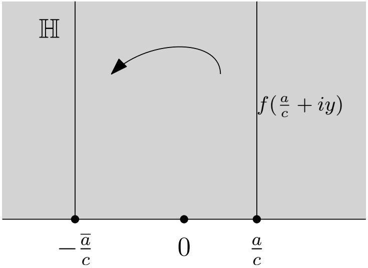
Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.



$$\int_0^{\infty} f(\tfrac{a}{c} + iy) y^s \frac{\mathrm{d}y}{y}$$

The Golfeld-Li proof

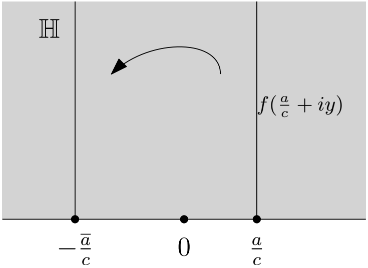
Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.



$$
\begin{aligned}
&\int_0^\infty f(\tfrac{a}{c} + iy) y^s \frac{\mathrm{d}y}{y} \\
&= \int_0^\infty \sum_{n=1}^{\infty} a(n) e^{2\pi i n \frac{a}{c}} e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} \\
&= (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{a(n) e\left(\frac{an}{c}\right)}{n^s}
\end{aligned}
$$

The Golfeld-Li proof

Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.

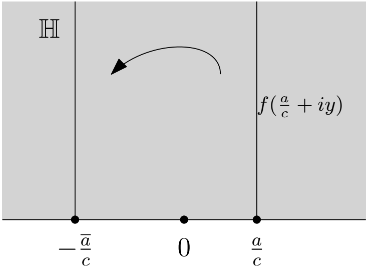


$$\int_0^\infty f(\tfrac{a}{c} + iy)y^s \frac{\mathrm{d}y}{y}$$
$$= \int_0^\infty \sum_{n=1}^{\infty} a(n)e^{2\pi i n \frac{a}{c}} e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y}$$
$$= (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)e\left(\frac{an}{c}\right)}{n^s}$$

Changing variables $y \to 1/y$ and making use of $\mathrm{SL}(2,\mathbb{Z})$-invariance,

The Golfeld-Li proof

Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.

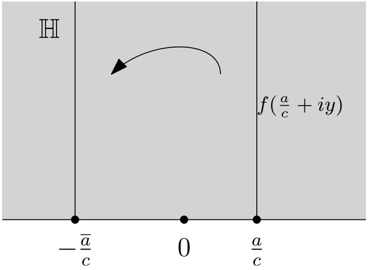


$$\int_0^\infty f(\tfrac{a}{c}+iy)y^s\frac{\mathrm{d}y}{y}$$
$$=\int_0^\infty \sum_{n=1}^{\infty} a(n)e^{2\pi i n\frac{a}{c}}e^{-2\pi n y}y^s\frac{\mathrm{d}y}{y}$$
$$=(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{a(n)e\left(\frac{an}{c}\right)}{n^s}$$

Changing variables $y \to 1/y$ and making use of $\mathrm{SL}(2,\mathbb{Z})$-invariance, the line of integration is transformed to another vertical line and we obtain a similar integral to what we started with,

The Golfeld-Li proof

Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.

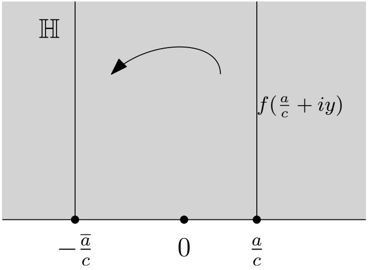


$$\begin{aligned}
&\int_0^{\infty} f(\tfrac{a}{c}+iy)y^s \frac{\mathrm{d}y}{y} \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} a(n)e^{2\pi i n \frac{a}{c}} e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} \\
&= (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)e\left(\frac{an}{c}\right)}{n^s}
\end{aligned}$$

Changing variables $y \to 1/y$ and making use of $\mathrm{SL}(2,\mathbb{Z})$-invariance, the line of integration is transformed to another vertical line and we obtain a similar integral to what we started with, save for $a \mapsto -\overline{a}$ and $s \mapsto 1-s$.

The Golfeld-Li proof

Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and consider the integral of $f(z)$ on the vertical line $\Re(z) = a/c$ against y^s.

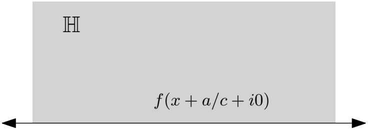


$$\int_0^\infty f(\tfrac{a}{c}+iy)y^s \frac{\mathrm{d}y}{y}$$
$$= \int_0^\infty \sum_{n=1}^{\infty} a(n)e^{2\pi i n \frac{a}{c}} e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y}$$
$$= (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)e\left(\frac{an}{c}\right)}{n^s}$$

Changing variables $y \to 1/y$ and making use of $\mathrm{SL}(2,\mathbb{Z})$-invariance, the line of integration is transformed to another vertical line and we obtain a similar integral to what we started with, save for $a \mapsto -\overline{a}$ and $s \mapsto 1-s$. A generalization of this "vertical integral" idea to $\mathrm{GL}(N)$ is the essence of the Goldfeld-Li proof.

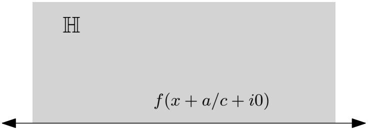
The Miller-Schmidt proof

In the Miller-Schmidt proof consider $f(z)$ "on the boundary of $\mathbb{H}$", against a Schwartz funciton g on $\mathbb{R}$.



$$\int_{-\infty}^{\infty} f(x + \tfrac{a}{c} + i0) g(x) \mathrm{d}x$$

In the Miller-Schmidt proof consider $f(z)$ "on the boundary of $\mathbb{H}$", against a Schwartz funciton g on $\mathbb{R}$.



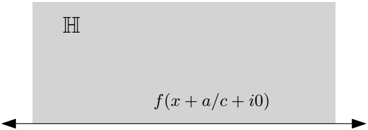
$$\int_{-\infty}^{\infty} f(x + \tfrac{a}{c} + i0) g(x) \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} a(n) e(\tfrac{na}{c}) e(nx) g(x) \mathrm{d}x$$

$$= \sum_{n=1}^{\infty} a(n) e\left(\tfrac{na}{c}\right) \widehat{g}(-n).$$

The Miller-Schmidt proof

In the Miller-Schmidt proof consider $f(z)$ "on the boundary of $\mathbb{H}$", against a Schwartz funciton g on $\mathbb{R}$.



$$\int_{-\infty}^{\infty} f(x + \tfrac{a}{c} + i0) g(x) \mathrm{d}x$$

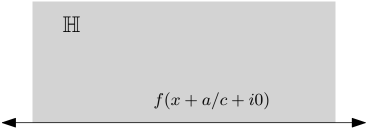
$$= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} a(n) e(\tfrac{na}{c}) e(nx) g(x) \mathrm{d}x$$

$$= \sum_{n=1}^{\infty} a(n) e\left(\tfrac{na}{c}\right) \widehat{g}(-n).$$

Once again, using the automorphy of f, the initial integral is transformed, and one obtains the other side of the Voronoi formula.

The Miller-Schmidt proof

In the Miller-Schmidt proof consider $f(z)$ "on the boundary of $\mathbb{H}$", against a Schwartz funciton g on $\mathbb{R}$.



$$\int_{-\infty}^{\infty} f(x + \tfrac{a}{c} + i0)g(x)\mathrm{d}x = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} a(n)e(\tfrac{na}{c})e(nx)g(x)\mathrm{d}x$$

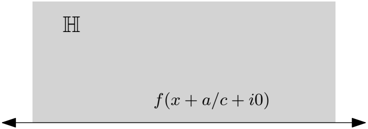
$$= \sum_{n=1}^{\infty} a(n)e\left(\tfrac{na}{c}\right)\widehat{g}(-n).$$

Once again, using the automorphy of f, the initial integral is transformed, and one obtains the other side of the Voronoi formula.

This does not make sense, the summations do not converge!

The Miller-Schmidt proof

In the Miller-Schmidt proof consider $f(z)$ "on the boundary of $\mathbb{H}$", against a Schwartz funciton g on $\mathbb{R}$.



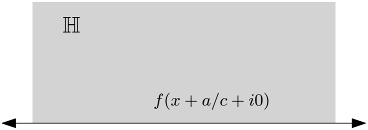
$$\int_{-\infty}^{\infty} f(x+\tfrac{a}{c}+i0)g(x)\mathrm{d}x = \int_{-\infty}^{\infty}\sum_{n=1}^{\infty} a(n)e(\tfrac{na}{c})e(nx)g(x)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} a(n)e\left(\tfrac{na}{c}\right)\widehat{g}(-n).$$

Once again, using the automorphy of f, the initial integral is transformed, and one obtains the other side of the Voronoi formula.
This does not make sense, the summations do not converge!That's why they had to develop the theory of automorphic distributions so that the computations made sense,

The Miller-Schmidt proof

In the Miller-Schmidt proof consider $f(z)$ "on the boundary of $\mathbb{H}$", against a Schwartz funciton g on $\mathbb{R}$.



$$\int_{-\infty}^{\infty} f(x + \tfrac{a}{c} + i0) g(x) \mathrm{d}x = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} a(n) e(\tfrac{na}{c}) e(nx) g(x) \mathrm{d}x$$

$$= \sum_{n=1}^{\infty} a(n) e\left(\tfrac{na}{c}\right) \widehat{g}(-n).$$

Once again, using the automorphy of f, the initial integral is transformed, and one obtains the other side of the Voronoi formula.
This does not make sense, the summations do not converge! That's why they had to develop the theory of automorphic distributions so that the computations made sense, and they were able to this in the generality of $\mathrm{GL}(N)$.

The double Dirichlet series proof

Take the Rankin-Selberg convolution of F on $\mathrm{GL}(N)$ and $E(z, s)$ on $\mathrm{GL}(2)$ is a double Dirichlet series.

The double Dirichlet series proof

Take the Rankin-Selberg convolution of F on $\mathrm{GL}(N)$ and $E(z, s)$ on $\mathrm{GL}(2)$ is a double Dirichlet series. The n^{th} Fourier coefficient of $E(z, s)$ is

The double Dirichlet series proof

Take the Rankin-Selberg convolution of F on $\mathrm{GL}(N)$ and $E(z, s)$ on $\mathrm{GL}(2)$ is a double Dirichlet series. The n^{th} Fourier coefficient of $E(z, s)$ is

$$\frac{\sigma_{1-2s}(n)}{\zeta(2s)} = \sum_{c=1}^{\infty} \frac{S(n, 0; c)}{c^{2s}}$$

The double Dirichlet series proof

Take the Rankin-Selberg convolution of F on $\mathrm{GL}(N)$ and $E(z,s)$ on $\mathrm{GL}(2)$ is a double Dirichlet series. The n^{th} Fourier coefficient of $E(z,s)$ is

$$\frac{\sigma_{1-2s}(n)}{\zeta(2s)} = \sum_{c=1}^{\infty} \frac{S(n,0;c)}{c^{2s}}$$

and therefore

$$\sum_{n=1}^{\infty} \frac{A(n)\sum_c S(n,0;c)c^{-2s}}{n^w} = \sum_c \frac{1}{c^{2s}} \sum_{\substack{a \pmod c \\ (a,c)=1}} \sum_{n=1}^{\infty} \frac{A(n)e(\frac{an}{c})}{n^w}$$

then the goal is to extract a single c and a single a modulo c.

The Double Dirichlet series

$$Z(s,w) = \frac{L(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})}$$

The Double Dirichlet series

$$Z(s,w) = \frac{L(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})}$$

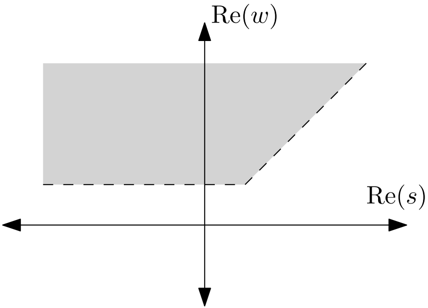
This is the Rankin Selberg convolution of $E(z,s,\chi^*)$ and F.

The Double Dirichlet series

$$Z(s,w) = \frac{L(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})}$$

This is the Rankin Selberg convolution of $E(z,s,\chi^*)$ and F.

It is a double Dirichlet series that converges in the shaded region in $\mathbb{C}^2$.



The Double Dirichlet series proof

Opening up the L-functions in $Z(s,w)$ we approximately get,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^w} \sum_{n=1}^{\infty} \frac{A(*,\ldots,*,n)g(\overline{\chi^*},\ell c^*,n)}{n^s}$$

The Double Dirichlet series proof

Opening up the L-functions in $Z(s,w)$ we approximately get,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^w} \sum_{n=1}^{\infty} \frac{A(*,\ldots,*,n)g(\overline{\chi^*},\ell c^*,n)}{n^s}$$

on the other hand the functional equation to the second L-function in the numerator sends it to $L(1-s,\widetilde{F}\times\chi^*)$.

The Double Dirichlet series proof

Opening up the L-functions in $Z(s, w)$ we approximately get,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^w} \sum_{n=1}^{\infty} \frac{A(*, \ldots, *, n) g(\overline{\chi^*}, \ell c^*, n)}{n^s}$$

on the other hand the functional equation to the second L-function in the numerator sends it to $L(1 - s, \widetilde{F} \times \chi^*)$.
Now opening up the Dirichlet series in sight, we obtain,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^w} \frac{G(s)}{c^{*Ns-1}} \sum_{n=1}^{\infty} \frac{A(n, *, \ldots, *) g(\chi^*, *, *)^{N-1}}{n^{1-s}}$$

The Double Dirichlet series proof

Opening up the L-functions in $Z(s,w)$ we approximately get,

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell^w}\sum_{n=1}^{\infty}\frac{A(*,\ldots,*,n)g(\overline{\chi^*},\ell c^*,n)}{n^s}$$

on the other hand the functional equation to the second L-function in the numerator sends it to $L(1-s,\widetilde{F}\times\chi^*)$.
Now opening up the Dirichlet series in sight, we obtain,

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell^w}\frac{G(s)}{{c^*}^{Ns-1}}\sum_{n=1}^{\infty}\frac{A(n,*,\ldots,*)g(\chi^*,*,*)^{N-1}}{n^{1-s}}$$

Then for each ℓ we get a functional equation of the Dirichlet series in the s variable.

The Double Dirichlet series proof

Opening up the L-functions in $Z(s,w)$ we approximately get,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^w} \sum_{n=1}^{\infty} \frac{A(*,\dots,*,n)g(\overline{\chi^*},\ell c^*,n)}{n^s}$$

on the other hand the functional equation to the second L-function in the numerator sends it to $L(1-s,\widetilde{F}\times\chi^*)$.
Now opening up the Dirichlet series in sight, we obtain,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell^w} \frac{G(s)}{c^{*Ns-1}} \sum_{n=1}^{\infty} \frac{A(n,*,\dots,*)g(\chi^*,*,*)^{N-1}}{n^{1-s}}$$

Then for each ℓ we get a functional equation of the Dirichlet series in the s variable.
Averaging over χ gives us the formula.

What's new?

- Using multiple Dirichlet series we were able to overcome technical difficulties of passing from functional equation of multiplicatively twisted to additively twisted Dirichlet series.

What's new?

- Using multiple Dirichlet series we were able to overcome technical difficulties of passing from functional equation of multiplicatively twisted to additively twisted Dirichlet series.
- Our proof does not use the automorphy of F, but only that there is a list of Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ which satisfy the Hecke relations, and when you form $L(s, F \times \chi)$ it has a functional equation.

What's new?

- Using multiple Dirichlet series we were able to overcome technical difficulties of passing from functional equation of multiplicatively twisted to additively twisted Dirichlet series.
- Our proof does not use the automorphy of F, but only that there is a list of Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ which satisfy the Hecke relations, and when you form $L(s, F \times \chi)$ it has a functional equation.
  - Therefore (for example) we can apply this to $F = G \times H$ or $F = G \oplus H$ where G and H are arbitrary Maass forms on $\mathrm{GL}(M_1)$ and $\mathrm{GL}(M_2)$

What's new?

- Using multiple Dirichlet series we were able to overcome technical difficulties of passing from functional equation of multiplicatively twisted to additively twisted Dirichlet series.
- Our proof does not use the automorphy of F, but only that there is a list of Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ which satisfy the Hecke relations, and when you form $L(s, F \times \chi)$ it has a functional equation.
  - Therefore (for example) we can apply this to $F = G \times H$ or $F = G \oplus H$ where G and H are arbitrary Maass forms on $\mathrm{GL}(M_1)$ and $\mathrm{GL}(M_2)$
- Our method of proof might be more easily generalizable to include level.

What's new?

- Using multiple Dirichlet series we were able to overcome technical difficulties of passing from functional equation of multiplicatively twisted to additively twisted Dirichlet series.
- Our proof does not use the automorphy of F, but only that there is a list of Fourier coefficients $A(m_1, m_2, \ldots, m_{N-1})$ which satisfy the Hecke relations, and when you form $L(s, F \times \chi)$ it has a functional equation.
  - Therefore (for example) we can apply this to $F = G \times H$ or $F = G \oplus H$ where G and H are arbitrary Maass forms on $\mathrm{GL}(M_1)$ and $\mathrm{GL}(M_2)$
- Our method of proof might be more easily generalizable to include level.
- The Rankin-Selberg convolution idea actually works for modular forms of half integer weight.